Math 5210 § 1. **Second Midterm Exam** Name: Solutions
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1. *State the Contraction Mapping Theorem. Let $X = \{\varphi \in \mathcal{C}(\mathbf{R}^n, \mathbf{R}^n) : \|\varphi\|_\infty < \infty\}$ be the space of continuous maps. X is complete under the sup-norm $\|\cdot\|_\infty$. Suppose for some $r \in (0,1)$ the map $a \in X$ satisfies $|a(x) - a(y)| \le r|x-y|$ for all $x, y \in \mathbf{R}^n$. Let $F(x) = x + a(x)$. Show that F has a unique inverse map for the form $G(y) = y + b(y)$ where $b \in X$. [Hint: Solve $I = F \circ G$ for b, where $I(y) = y$ is the identity map.]*

Contraction Mapping Theorem. *Let (X, d) be a complete metric space and $f : X \to X$ be a map such that for some $r \in (0,1)$,*

$$d(f(x) - f(y)) \le r d(x,y) \qquad \text{for all } x, y \in X.$$

Then there is a unique fixed point $z \in X$ such that $z = f(z)$.

We argue that $F(x) = x + a(x)$ is invertible. It is injective because for any $x, y \in \mathbf{R}^n$ we have

$$|F(x) - F(y)| = |x + a(x) - y - a(y)| \ge |x-y| - |a(x) - a(y)| \ge |x-y| - r|x-y| = (1-r)|x-y|$$

Thus $F(x) = F(y)$ implies $x = y$ so F is injective.

Now we show that F is surjective by constructing a right inverse of the form $G(y) = y + b(y)$ where $b \in X$. Thus we substitute in the equation $I = F \circ G$ to get

$$y = F(G(y)) = G(y) + a(G(y)) = y + b(y) + a(y + b(y))$$

or

$$b(y) = -a(y + b(y)) = T[b](y).$$

The solution is a fixed point of the Nemitskii operator T. If $b \in X$ then $T[b]$ is continuous because it is the composition of continuous functions and $T : X \to X$ because for all $y \in \mathbf{R}^n$,

$$|T[b](y)| \le \|a\|_\infty$$

so $T[b]$ is bounded. We claim that T is a contraction on X. Indeed,

$$\begin{aligned} |T[b_1](y) - T[b_2](y)| &= |-a(y + b_1(y)) + a(y + b_2(y))| \le r|y + b_1(y) - y - b_2(y)| \\ &= r|b_1(y) - b_2(y)| \le r\|b_1 - b_2\|_\infty. \end{aligned}$$

Taking the supermum over $y \in \mathbf{R}^n$ gives

$$\|T[b_1] - T[b_2]\|_\infty \le r\|b_1 - b_2\|_\infty,$$

so T is a contraction on the complete space X. Hence it has a unique fixed point $b \in X$ and $G(y) = y + b(y)$ exists. This proves the map F is surjective. Indeed, for any $y \in \mathbf{R}^n$ we have $x = G(y) \in \mathbf{R}^n$ so that $F(x) = F(G(y)) = y$. Thus F is surjective and so there is an inverse map $F^{-1} : \mathbf{R}^n \to \mathbf{R}^n$. Finally, we check that G is the inverse. Indeed, for every y we have

$$F(F^{-1}(y)) = y = F(G(y)).$$

Since F is injective, $F^{-1}(y) = G(y)$.

Though not part of the proof, here is how to see that X is complete. Suppose $\{f_n\} \subset X$ is a Cauchy sequence. Then for all $\epsilon > 0$ there is $N \in \mathbb{N}$ such that

$$\|f_m - f_n\|_\infty < \epsilon \qquad \text{whenever } m, n \ge N.$$

This implies that at any point $x \in \mathbf{R}^n$,

$$|f_m(x) - f_n(x)| < \epsilon \qquad \text{whenever } m, n \geq N.$$

Hence $\{f_n(x)\}$ is a Cauchy sequence in $\mathbf{R}^n$, thus convergent $f_n(x) \to f(x)$ as $n \to \infty$. As f_n is continuous and the convergence $f_n \to f$ is uniform in $\mathbf{R}^n$, we conclude that f is continuous. Finally, taking $\epsilon = 1$ we fix n sufficiently large so that $\|f_n - f\| < \epsilon$ which implies

$$\|f\|_\infty = \|(f - f_n) + f_n\|_\infty \leq \|f - f_n\|_\infty + \|f_n\|_\infty \leq 1 + \|f_n\|_\infty < \infty$$

so f is bounded.

2. *For a continuous, 2π-periodic function f, the Nth Fourier polynomial is*

$$S_N f(x) = \sum_{-N}^{N} c_k e^{ikx}, \qquad \text{where} \qquad c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-ikt}\, dt.$$

Define the Dirichlet Kernel, $D_N(t)$. Write $S_N f(x)$ in terms of $D_N(t)$. Assume that f is a continuously differentiable 2π-periodic function which in a $\delta > 0$ neighborhood of $x_0 \in \mathbf{R}$ equals for some constants α and β,

$$f(t) = \alpha + \beta \sin\left(\frac{t - x_0}{2}\right) \qquad \text{if } |x_0 - t| < \delta.$$

Show that $S_N f(x_0) \to f(x_0)$ as $N \to \infty$.

The Dirichlet kernel is

$$D_N(t) = \sum_{k=-N}^{N} e^{ikt} = \frac{\sin\left(N + \frac{1}{2}\right) t}{\sin\left(\frac{t}{2}\right)}.$$

Thus the Fourier polynomial

$$S_N f(x) = \sum_{k=-N}^{N} c_k e^{ikx} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x - t) D_N(t)\, dt.$$

Let x_0 and $\delta > 0$ be as in the problem. Then

$$\begin{aligned}
S_N f(x_0) - f(x_0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(x_0 - t) - f(x_0)] D_N(t)\, dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \sin\left(N + \frac{1}{2}\right) t\, dt \\
&= \frac{1}{2\pi} \left(\int_{|t|<\delta} + \int_{t=\delta}^{\pi} + \int_{t=-\pi}^{-\delta} \right) g(t) \sin\left(N + \frac{1}{2}\right) t\, dt \\
&= I + II + III,
\end{aligned}$$

where

$$g(t) = \frac{f(x_0 - t) - f(x_0)}{\sin\left(\frac{t}{2}\right)}.$$

When $|t| < \delta$ we have

$$g(t) = \frac{\alpha + \beta \sin\left(\dfrac{(x_0 - t) - x_0}{2}\right) - \alpha - \beta \sin\left(\dfrac{x_0 - x_0}{2}\right)}{\sin\left(\dfrac{t}{2}\right)} = -\beta$$

so that

$$|I| = \left| -\frac{\beta}{2\pi} \int_{|t|<\delta} \sin\left(N + \frac{1}{2}\right) t \, dt \right| = 0$$

because the integrand is an odd function. When $|t| \geq \delta$ we don't divide by zero so g is continuously differentiable and

$$g'(t) = \frac{-f'(x_0 - t)}{\sin\left(\dfrac{t}{2}\right)} - \frac{[f(x_0 - t) - f(x_0)] \cos\left(\dfrac{t}{2}\right)}{2 \sin^2\left(\dfrac{t}{2}\right)}.$$

By the mean value theorem $f(x_0 - t) - f(x_0) = -f'(c)t$ where $c(t)$ is between x_0 and $x_0 - t$

$$g(t) = \frac{-f'(c)t}{\sin\left(\dfrac{t}{2}\right)}$$

$$g'(t) = \frac{-f'(x_0 - t)}{\sin\left(\dfrac{t}{2}\right)} + \frac{f'(c)t \cos\left(\dfrac{t}{2}\right)}{2 \sin^2\left(\dfrac{t}{2}\right)}$$

Using $\sin t/2 \geq t/\pi$ is increasing on $[\delta, \pi]$,

$$|g(t)| \leq \pi \|f'\|$$

$$|g'(t)| \leq \frac{\|f'\|}{\sin\left(\dfrac{\delta}{2}\right)} + \frac{\pi \|f'\|}{2 \sin\left(\dfrac{\delta}{2}\right)}$$

Integrating by parts,

$$II = \frac{1}{2\pi} \int_{t=\delta}^{\pi} g'(t) \frac{\cos\left(N + \dfrac{1}{2}\right) t}{N + \frac{1}{2}} \, dt + \frac{1}{2\pi} \left[g(t) \frac{\cos\left(N + \dfrac{1}{2}\right) t}{N + \frac{1}{2}} \right]_{t=\delta}^{\pi}$$

With a similar estimate for III,

$$|II| + |III| \leq \frac{\left(1 + \dfrac{\pi}{2}\right) \|f'\|}{\sin\left(\dfrac{\delta}{2}\right) (N + \frac{1}{2})} + \frac{\|f'\|}{N + \frac{1}{2}}$$

Hence

$$|S_N f(x_0) - f(x_0)| \leq |I| + |II| + |III| = \mathbf{O}\left(\frac{1}{N}\right) \qquad \text{as } N \to \infty.$$

Alternately, we may use the Riemann-Lebesgue Lemma. Since g is continuous on $[\delta, \pi]$, we see that

$$II = \frac{1}{2\pi} \int_\delta^\pi g(t) \sin\left(N + \frac{1}{2}\right) t\, dt \to 0 \qquad \text{as } N \to \infty.$$

Similarly for III. Thus $S_N f(x_0) \to f(x_0)$ but at an unknown rate as $N \to \infty$.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample. In all problems, let (X, d) be a complete metric space.*

   (a) *Let $(\mathcal{K}(\mathbf{R}^n), h)$ be the compact subsets of $\mathbf{R}^n$ with Hausdorff metric. Let $\{c_n\} \subset \mathbf{R}^n$ be a sequence such that $c_n \to c$ and $\{r_n\}$ be a real sequence such that $r_n \ge 0$ and $r_n \to r$. Then the sequence of closed balls $\bar{B}_n \subset \mathbf{R}^n$ with center c_n and radius r_n converges in $(\mathcal{K}(\mathbf{R}^n), h)$.*

   True. Observe that because $\bar{B}(c_\ell, r_\ell) \subset (\bar{B}(c_\ell, r_k))_{|r_\ell - r_k|} \subset ((\bar{B}(c_k, r_k))_{|c_\ell - c_k|})_{|r_\ell - r_k|} = (\bar{B}(c_k, r_k))_{|c_\ell - c_k| + |r_\ell - r_k|}$, each ball is in the $|c_\ell - c_k| + |r_\ell - r_k|$ collar of the other

$$\bar{B}_\ell \subset (\bar{B}_k)_{|c_\ell - c_k| + |r_\ell - r_k|}, \qquad \bar{B}_k \subset (\bar{B}_\ell)_{|c_\ell - c_k| + |r_\ell - r_k|}$$

   so $h(\bar{B}_k, \bar{B}_\ell) \le |c_\ell - c_k| + |r_\ell - r_k|$. Hence $\{\bar{B}_k\}$ is a Cauchy sequence in $(\mathcal{K}(\mathbf{R}^n), h)$ because $\{c_k\} \subset \mathbf{R}^n$ and $\{r_k\} \subset \mathbf{R}$ are Cauchy sequences. But $(\mathcal{K}(\mathbf{R}^n), h)$ is complete so $\{\bar{B}_k\}$ is convergent.

   (b) *Suppose $f : X \to X$ satisfies $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$ such that $x \ne y$. Then f has a fixed point: there is $z \in X$ such that $z = f(z)$.*

   False. The map is not assumed to be r-Lipschitz for some $r \in (0, 1)$ so the Contraction Mapping Theorem does not apply. Any differentiable function $f : \mathbf{R} \to \mathbf{R}$ will provide a counterexample provided $f(x) > x$ for all $x \in \mathbf{R}$ so f has no fixed point and provided $|f'(x)| < 1$ everywhere so by the Mean Value Theorem, if $x \ne y$ then $|f(x) - f(y)| = |f'(c)||x - y| < |x - y|$ where c is some number between x and y. An example is

$$f(x) = \sqrt{1 + x^2}.$$

   (c) *Let $\mathcal{A} \subset \mathcal{C}(M)$ be an algebra of continuous real functions on compact $M \subset X$ such that for every $x, y \in M$ there is $g \in \mathcal{A}$ such that $g(x) \ne g(y)$. Then any continuous function on M may be uniformly approximated by functions of $\mathcal{A}$.*

   False. The algebra also needs functions that are nonzero at any given point for the Stone-Weierstrass Theorem to apply. So for example, consider the polynomials without constant term $\mathcal{A} = \{\sum_{k=1}^n c_k x^k : c_k \in \mathbf{R},\ n \in \mathbb{N}\}$ on $[-1, 1]$. Then $f(x) = 1$ cannot be approximated in sup-norm because for every $\psi \in \mathcal{A}$ we have $\|f - \psi\|_\infty \ge |f(0) - \psi(0)| = 1$.

4. *The Fourier Series for the sawtooth function $\varphi(x) = |x|$ on the interval $|x| \le \pi$ and extended to $\mathbf{R}$ by periodicity of period 2π is*

$$\varphi(x) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^\infty \frac{\cos(2k-1)x}{(2k-1)^2}. \tag{1}$$

   Let $X = \mathcal{C}_{per}([-\pi, \pi])$ be the 2π-periodic continuous functions on $\mathbf{R}$. Define the usual complex inner product $\langle f, g\rangle$ and corresponding distance $\|f - g\|$ on X. Let $V_n = \{\sum_{-n}^n c_k e^{ikx} : c_k \in \mathbf{C}\}$ be the linear subspace of trig polynomials of degree n. Let $P_{V_n} : X \to V_n$ be the orthogonal projection to V_n. Find $P_{V_n}[\varphi]$ and explain. What is the distance $\|\varphi - P_{V_n}[\varphi]\|$? Do the $P_{V_n}[\varphi]$ approximate φ in $\|\bullet\|$?

The inner product and norm on X are

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\bar{g}(t)\,dt, \qquad \|f\| = \sqrt{\langle f, f\rangle}.$$

The $\psi_k(x) = e^{ikx}$ for $k \in \mathbb{Z}$ are an orthonormal set of functions. The projection to $V_N = \operatorname{span}\{e^{-iNx}, e^{-i(N-1)x}, \dots, e^{iNx}\}$ is given by the Fourier polynomial

$$P_N(f) = \sum_{k=-N}^{N} \langle f, \psi_k\rangle \psi_k(x) = \sum_{k=-N}^{N} c_k e^{ikx} = S_N f(x)$$

where

$$c_k = \langle f, \psi_k\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}\,dt.$$

In the present case, for $N \ge 1$ and using $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$,

$$S_{2N}\varphi(x) = S_{2N-1}\varphi(x) = \frac{2\pi}{2} - \frac{2}{\pi}\sum_{k=1}^{N} \frac{1}{(2k-1)^2}\left(e^{i(2k-1)x} + e^{-i(2k-1)x}\right).$$

so for $k > 0$,

$$c_0 = \frac{\pi}{2}, \qquad c_{\pm(2k-1)} = -\frac{2}{\pi(2k-1)^2}, \qquad c_{\pm 2k} = 0.$$

We know that $\{\psi_k\}$ is a complete orthonormal system because of Fejér's theorem, that any continuous function can be uniformly approximated by trig polynomials, hence also in the $\mathcal{L}^2$ sense. It follows that Parseval's formula holds

$$\sum_{k=-\infty}^{\infty} |c_k|^2 = \|\psi\|^2.$$

However, we also know that the norm of the projection

$$\|P_n\varphi\|^2 = \|S_n\varphi\|^2 = \sum_{-n}^{n} |c_k|^2$$

Using the orthogonality of $P_n\varphi$ and $\varphi - P_n\varphi$ we see that

$$\|\varphi - P_n\varphi\|^2 = \|\varphi\|^2 - \|P_n\varphi\|^2 = \sum_{|k|>n} |c_k|^2 \to 0$$

as $n \to \infty$.

5. *Another summability kernel for Fourier Series, due to Dunham Jackson is*

$$J_n(t) = \frac{3}{2(n+1)(2n^2+4n+3)}\left(\frac{\sin\dfrac{(n+1)t}{2}}{\sin\dfrac{t}{2}}\right)^4 = \sum_{k=-2n}^{2n} a_{2n,k}e^{ikt}$$

which can be expressed as a trigonometric sum of degree $2n$.

J_n has the three properties of an approximate identity on $[-\pi, \pi]$. State the third property and show that J_N satisfies it.

(a) $J_n(x) \ge 0$.

(b) $\int_{-\pi}^{\pi} J_n(x)\,dx = 2\pi$.

(c) $\boxed{\text{For any } \delta > 0, \displaystyle\int_{\delta \le |x| \le \pi} J_n(x)\,dx \to 0 \text{ as } n \to \infty.}$

Because $\sin t/2$ is increasing, for $\delta \le |x| \le \pi$,

$$J_n(x) \le \frac{3}{2(n+1)(2n^2+4n+3)\sin^4\left(\frac{\delta}{2}\right)}$$

so

$$0 \le \int_{\delta \le |x| \le \pi} J_n(x)\,dx \le \frac{3\pi}{(n+1)(2n^2+4n+3)\sin^4\left(\frac{\delta}{2}\right)} \to 0 \quad \text{as } n \to \infty.$$

Show using J_n that there if f is a continuous 2π-periodic function, then there is a sequence of trigonometric polynomials such that

$$\|f - p_n\|_\infty \to 0, \qquad \text{as } n \to \infty.$$

Because $J_n(t)$ is trigonometric, the convolution

$$p_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} J_n(x-t)f(t)\,dt$$

is a trigonometric polynomial of degree $2n$. Also, f is uniformly continuous at all x so for any $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ so that

$$|f(x) - f(y)| < \frac{\varepsilon}{2} \qquad \text{whenever } x, y \in \mathbf{R} \text{ satisfy } |x-y| < \delta.$$

For any $\epsilon > 0$ we let $\delta = \delta(\varepsilon)$ as above take N so large that

$$\int_{\delta \le |x| \le \pi} J_n(x)\,dx < \frac{\pi\epsilon}{2\|f\|_\infty + 1} \quad \text{whenever } n \ge N.$$

Now if $n \ge N$ we estimate at any x using (b) and (c),

$$\begin{aligned}
|p_n(x) - f(x)| &\le \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x-t) - f(x)]J_n(t)\,dt\right| \\
&\le \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x-t) - f(x)|J_n(t)\,dt \\
&= \frac{1}{2\pi}\int_{|t|\le\delta}|f(x-t) - f(x)|J_n(t)\,dt + \frac{1}{2\pi}\int_{\delta\le|t|\le\pi}|f(x-t) - f(x)|J_n(t)\,dt \\
&\le \frac{\varepsilon}{4\pi}\int_{|t|\le\delta} J_n(t)\,dt + \frac{\|f\|_\infty}{\pi}\int_{\delta\le|t|\le\pi} J_n(t)\,dt \\
&\le \frac{\varepsilon}{4\pi}\int_{-\pi}^{\pi} J_n(t)\,dt + \frac{\varepsilon\|f\|_\infty}{2\|f\|_\infty + 1} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus $p_n \to f$ uniformly as $n \to \infty$.