Homework for Math 5210 §1, Spring 2017

A. Treibergs, Instructor

April 14, 2017

Our text is by Robert S. Strichartz, *The Way of of Analysis,* revised edition, Jones and Bartlett Publishers, Sudbury (2000). Please read the relevant sections in the text as well as any cited reference. Assignments are due the following Friday, or on April 21, whichever comes first.

Your written work reflects your professionalism. Make answers complete and self contained. This means that you should copy or paraphrase each question, provide adequate explanation to help the reader understand the structure of your argument, be thorough in the details, state any theorem that you use and proofread your answer.

Homework from Wednesday to Tuesday will be due Friday. Late homework that is up to one week late will receive half credit. Homework that is more than one week late will receive no credit at all. Homework that is placed in my mailbox in JWB 228 before 4 pm Friday afternoon will be considered to be on time.

Please hand in problems A1 – A2 on Friday, January 13.

A1. Countable/Uncountable. Please hand in these exercises from Strichartz's *The Way of Analysis.*

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13[2, 5]
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A2. Bijection. Exhibit an explicit bijection between $(0, 1)$ and $[0, 1]$. (*Cf.* R. Gariepy & W. Ziemer, *Modern Real Analysis,* PWS Publishing, Boston, 1995, p. 33.)

Please hand in problems B on Friday, January 20.

B. Construction of the Reals. Please hand in these exercises from Strichartz's *The Way of Analysis.*

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48[1, 3, 5, 8ad, 9c]
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Please hand in problems C on Friday, January 27.

C. Completeness. Please hand in these exercises from Strichartz's *The Way of Analysis.*

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54[2, 3, 9]
84[1, 5, 9]
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Please hand in problems D on Friday, February 3.

D. Topology of $\mathbb{R}^1$. Please hand in these exercises from Strichartz's *The Way of Analysis.*

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98[1, 2, 7, 8, 10, 11]
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Please hand in problems E1–E3 on Friday, February 10.

E1. Compactness. If (X, d) is a compact metric space and $F_n \subset X$ are nonempty closed subsets such that $F_n \supset F_{n+1}$ for all n, then $\cap_{n=1}^{\infty} F_n$ is nonempty.

E2. Equicontinuity. If (X, d) is a compact metric space and $E \subset \mathcal{C}(X)$ is an equicontinuous family of functions, then the closure $\overline{E}$ is equicontinuous.

E3. Metric Spaces. Please hand in these exercises from Strichartz's *The Way of Analysis.*

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384[2  ,3 ,10]
409[1, 20]
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Please hand in problems F on Friday, February 17.

F. Connectedness and Contractions. Please hand in these exercises from Strichartz's *The Way of Analysis.*

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314[4, 6]
409[7, 16, 17, 26]
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Please hand in problems G1–G5 on Friday, February 24.

G1. Inverse Maps. Suppose $U \subset \mathbf{R}^n$ is a convex open set and $F : U \to \mathbf{R}^n$ a $\mathcal{C}^1$ function such that $dF(x)$ is positive definite at every point of U. Show that then F is one-to-one. Show by example this fails if $U \subset \mathbf{R}^2$ is only connected rather than convex. [J. Taylor, *Foundations of Analysis,* p. 265.]

G2. Fredholm Integal Equation. Let $I = [a, b]$, $g : I \to \mathbf{R}^n$ and $K : I \times I \to M_{n\times n}(\mathbf{R})$ be continuous functions, where $M_{n\times n}(\mathbf{R})$ are real $n \times n$ matrices. Find $\lambda_0 > 0$ so that if $|\lambda| \le \lambda_0$, then there is a unique continuous function $x : I \to \mathbf{R}^n$ that solves the Fredholm Integral Equation. [Haaser & Sullivan, *Real Analysis,* p. 104.]

$$x(t) = \lambda \int_a^b K(t, s)x(s)\, ds + g(t).$$

G3. Hausdorff Metric. Find $d_A(B)$, $d_B(A)$ and $h(A, B)$, the Hausdorff distance for the given A and B.

a. $A = \{(x, y) : 0 \le x \le 1 \text{ and } 0 \le y \le 1\}$, $\quad B = \{(x, y) : x^2 + y^2 \le 1\}$;

b. $A = \{(x, y) : |x| \le 1 \text{ and } |y| \le 1\}$, $\quad B = \{(x, y) : x^2 + y^2 \le 1\}$;

c. $A = \{(x, y) : |x| \le 1 \text{ and } |y| \le 1\}$, $\quad B = \{(x, y) : x^2 + y^2 \le 2\}$;

G4. Enlarged Sets. Let $A, B, C, D \in \mathcal{K}(\mathbf{R}^n)$.

a. For $\epsilon > 0$ show $h(A_\epsilon, B_\epsilon) \le h(A, B)$.

b. Let $A \boxplus B = \{a + b : a \in A,\ b \in B\}$ be the Minkowski sum. Show $h(A \boxplus B, C \boxplus D) \le h(A, C) + h(B, D)$.
[Hadwiger, *Vorlesungen Über Inhalt, Oberfläche und Isoperimetrie,* p. 152.]

G5. Decreasing Sequence. Let $K_n \in \mathcal{K}(\mathbf{R}^n)$ such that $K_n \supset K_{n+1}$ for all n. Show that in $(\mathcal{K}(\mathbf{R}^n), h)$,

$$\lim_{n\to\infty} K_n = K_\infty \qquad \text{where} \qquad K_\infty = \bigcap_{n=1}^{\infty} K_n.$$

[D. Burago, Y. Burago & S. Ivanov, *A Course in Metric Geometry.* p. 253.]

Please hand in problems H on Friday, March 3.

H. Weierstrass Approximation Theorem. Please hand in these exercises from Strichartz's *The Way of Analysis.*

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307[3, 4, 10]
409[12, 13, 18]
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Please hand in problems I1–I2 on Friday, March 10.

I1. Differentiating a Series. Show that if f_n are differentiable functions on $[a, b]$, $\sum_{n=1}^{\infty} f_n(c)$ converges for some $c \in [a, b]$ and $\sum_{n=1}^{\infty} {f_n}'$ converges uniformly on $[a, b]$ then $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[a, b]$ and $f'(x) = \sum_{n=1}^{\infty} {f_n}'(x)$ for all $x \in [a, b]$.

I2. Fourier Series. Please hand in these exercises from Strichartz's *The Way of Analysis.*

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528[1, 4, 6, 13]
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Please hand in problems J on Friday, March 24.

J. Summability Methods. Please hand in these exercises from Strichartz's *The Way of Analysis.*

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559[2, 3, 6, 12, 14]
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Please hand in problems K1–K2 on Friday, March 31.

K1. Riemann-Lebesgue Lemma. If $f \in \mathcal{C}([a, b])$ then $\displaystyle\lim_{n\to\infty} \int_a^b f(t) e^{int}\, dt = 0.$

K2. Poisson Kernel. Please hand in this exercize from Strichartz's *The Way of Analysis.*

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559[16]
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Please hand in problems L1–L2 on Friday, April 7.

L1. Borel Sets. If $A \subset \mathbf{R}$ and $\epsilon > 0$ there is an open set U such that $A \subset U$ and $m^*U \le m^*A + \epsilon$ and a $\mathcal{G}_\delta$ set G such that $A \subset G$ and $m^*A = m^*G$. [H. Royden, *Real Analysis*, 2nd. ed., Macmillan 1968, p. 56.]

L2. Outer Measure. Please hand in these exercises from Strichartz's *The Way of Analysis.*

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641[5, 6, 10, 16]
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Please hand in problems M1–M5 on Friday, April 14.

M1. Decreasing Sequence. Show that the condition $mE_1 < \infty$ is necessary in Proposition 14 by giving a decreasing sequence $\{E_i\}$ of measurable sets with $\emptyset = \cap E_n$ and $mE_n = \infty$ for each n. [H. Royden, *Real Analysis*, 2nd. ed., Macmillan 1968, p. 62[11].]

M2. Decreasing Again. Give an example of a decreasing sequence $\{E_i\}$ of sets with $m^*E_n < \infty$ and $m^*(\cap E_n) < \lim_{n\to\infty} m^*E_n$. [H. Royden, *Real Analysis*, 2nd. ed., Macmillan 1968, p. 65[17b].]

M3. Products of Measurable Functions.

a. Let f be an extended real-valued function with measurable domain D and let $D_1 = \{x \in D : f(x) = \infty\}$ and $D_2 = \{x \in D : f(x) = -\infty\}$. Then f is measurable if and only if D_1 and D_2 are measurable and the restriction of f to $D \backslash (D_1 \cup D_2)$ is measurable.

b. Prove that the product of two measurable extended real-valued functions is measurable.

c. If f and g are measurable extended real-valued functions with measurable domains and α a fixed number, then $f + g$ is measurable provided that we define $f + g$ to be α whenever it is of the form $\infty - \infty$ or $-\infty + \infty$.

d. Let f and g be measurable extended real-valued functions which are finite almost everywhere. Then $f + g$ is measurable no matter how it is defined at points where it has the form $\infty - \infty$.

[H. Royden, *Real Analysis*, 2nd. ed., Macmillan 1968, p. 69[22].]

M4. Pullback of a Borel Set. Let f be measurable and B a Borel set. Then $f^{-1}[B]$ is a measurable set. [Hint: The class of sets for which $f^{-1}[E]$ is measurable is a σ-algebra.] [H. Royden, *Real Analysis*, 2nd. ed., Macmillan 1968, p. 70[24].]

M5. Composition. Show that if f is a measurable real valued function and g a continuous function defined on $(-\infty, \infty)$ then $g \circ f$ is measurable. [H. Royden, *Real Analysis*, 2nd. ed., Macmillan 1968, p. 70[25].]

Please hand in problems N1–N6 on Friday, April 21.

N1. Zero Integral. Let f be a nonnegative measurable function. Show that $\int f = 0$ implies $f = 0$ *a.e.* [H. Royden, *Real Analysis*, 2nd. ed., Macmillan 1968, p. 86[3].]

N2. Indefinite Integral. Let f be a nonnegative integrable function. Show that the function $F(x)$ defined by

$$F(x) = \int_{-\infty}^{x} f(x)\, dx$$

is continuous using the Monotone Convergence Theorem. [H. Royden, *Real Analysis*, 2nd. ed., Macmillan 1968, p. 86[5].]

N3. Fatou's Lemma and MCT. Show that we can have strict inequality in Fatou's Lemma. Hint: consider the sequence $\{f_n\}$ where $f_n(x) = 1$ if $n \le x \le n+1$ and $f_n(x) = 0$ otherwise.

Show that the Monotone Convergence Theorem need not hold for decreasing sequences of functions. Hint: consider the sequence $\{g_n\}$ where $g_n(x) = 0$ if $x < n$ with $g_n(x) = 1$ otherwise. [H. Royden, *Real Analysis*, 2nd. ed., Macmillan 1968, p. 86[7].]

N4. Convergence on Subsets. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on $(-\infty, \infty)$ such that $f_n \to f$ *a.e.* and suppose that $\int f_n \to \int f < \infty$, Show that for each measurable set E we have $\int_E f_n \to \int_E f$. [H. Royden, *Real Analysis*, 2nd. ed., Macmillan 1968, p. 86[9].]

N5. $\mathcal{L}^1$ Convergence. Let $\{f_n\}$ be a sequence of integrable functions such that $f_n \to f$ *a.e.* with f integrable. Then $\int |f_n - f| \to 0$ if and only if $\int |f_n| \to \int |f|$. [H. Royden, *Real Analysis*, 2nd. ed., Macmillan 1968, p. 90[14].]

N6. Riemann-Lebesgue Lemma. Let f be an integrable function on $(-\infty, \infty)$. Show that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) \cos nx \, dx = 0.$$

Hint: first show that for every $\varepsilon > 0$ there is a step function ψ such that

$$\int_{-\infty}^{\infty} |f - \psi| < \varepsilon.$$

[H. Royden, *Real Analysis*, 2nd. ed., Macmillan 1968, p. 90[16].]

The FINAL EXAM is Mon., May 1 at 1:00 - 3:00 PM in the usual classroom, LCB 219.