Math 5210 § 1.	**First Midterm Exam**	Name: Solutions
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1. *Let $A \subset \mathbf{R}$ be a nonempty subset. Define: A is* sequentially compact *(what our author calls compact.) Define: A is* closed. *Using just your definitions and elementary facts about sequences, but without quoting results from the text or elsewhere, show that if A is sequentially compact, then it is closed.*

   A set $A \subset \mathbf{R}$ is sequentially compact if every sequence $\{x_i\}$ in A has a limit point $a \in A$. That is, for every $\epsilon > 0$ there is an infinite number of terms x_i that satisfy $|x_i - a| < \epsilon$.

   A set $A \subset \mathbf{R}$ is closed if it contains all of its limit points. x is a limit point (cluster point) of A if for any $\epsilon > 0$ there is $y \in A$ not equal to x such that $|x - y| < \epsilon$.

   Suppose y is a limit point of A, to show $y \in A$. This means for every $n \in \mathbb{N}$ there is $y_n \in A$ so that $0 < |y_n - y| < 1/n$. Thus $\{y_n\}$ is a sequence in A which converges to y. By sequential compactness, the sequence $\{y_n\}$ has a limit point $a \in A$. But since the sequence is converging, there is only one limit point, so $y = a$ which is a point in A. Thus A contains its limit points, thus is closed.

2. *Let $(X, \langle\cdot,\cdot\rangle)$ be a real inner product space. Let $v \in X$ be any unit vector. Let $\mathcal{H} = \{x \in X : \langle x, v\rangle > 0\}$. For $x, y \in X$ describe the norm $\|x\|$ and distance $d(x, y)$ associated to the inner product $\langle\cdot,\cdot\rangle$. Define: $\mathcal{H}$ is open. Show that $\mathcal{H}$ is open. Define: x is a* limit point *(same as cluster point) of $\mathcal{H}$. Determine the limit points of $\mathcal{H}$ and prove your result.*

   The norm and distance are given by $\|x\| = \sqrt{\langle x, x\rangle}$ and $d(x, y) = \|x - y\|$.

   $\mathcal{H}$ is open if for every point $x \in \mathcal{H}$ there is an $r > 0$ so that the ball $B_r(x) \subset \mathcal{H}$, where $B_r(x) = \{y \in X : \|x - y\| < r\}$.

   So see that $\mathcal{H}$ is open, we choose $x \in \mathcal{H}$ and show that for $r = \langle x, v\rangle > 0$ we have $B_r(x) \subset \mathcal{H}$. To see it, pick $z \in B_r(x)$ so $\|x - z\| < r$. Then, using the Cauch Schwartz inequality and $\|v\| = 1$,

$$\begin{aligned}
\langle z, v\rangle &= \langle x + (z - x), v\rangle \\
&= \langle x, v\rangle + \langle z - x, v\rangle \\
&\ge \langle x, v\rangle - |\langle z - x, v\rangle| \\
&\ge \langle x, v\rangle - \|z - x\|\,\|v\| \\
&> r - r \cdot 1 = 0.
\end{aligned}$$

   Thus $z \in \mathcal{H}$ so $B_r(x) \subset \mathcal{H}$.

   x is a limit point (cluster point) of $\mathcal{H}$ if for any $\epsilon > 0$ there is $y \in \mathcal{H}$ not equal to x such that $\|x - y\| < \epsilon$.

   The limit points of $\mathcal{H}$ is the set $\mathcal{L} = \{x \in X : \langle x, v\rangle \ge 0\}$. To see all points in $z \in \mathcal{L}$ are limit points, consider the sequences

$$z_n = z + \frac{1}{2n} v$$

   $z_n \in \mathcal{H}$ because

$$\begin{aligned}
\langle z_n, v\rangle &= \left\langle z + \frac{1}{2n} v, v \right\rangle \\
&= \langle z, v\rangle + \frac{1}{2n}\langle v, v\rangle \\
&= 0 + \frac{1}{2n} \cdot 1 > 0.
\end{aligned}$$

For any $n \in \mathbb{N}$ we have

$$\|z - z_n\| = \left\| \frac{1}{2n} v \right\| = \frac{1}{2n} \leq \frac{1}{n}$$

so z is a limit point.

To see that no other points are limit points, we suppose $w \notin \mathcal{L}$ or $\langle x, v \rangle < 0$. But this says $\langle x, -v \rangle > 0$. Using $-v$ as the unit vector instead of v, we showed that $\mathcal{L}' = \{x \in X : \langle x, -v \rangle > 0\}$ is open, so there is $r > 0$ so that $B_r(w) \subset \mathcal{L}'$. Now $\mathcal{H} \subset \mathcal{L}$ so in particular, $B_r(w) \bigcap \mathcal{H} = \emptyset$ so w is not a limit point of $\mathcal{H}$.

3. *The real numbers were defined to be equivalence classes $\mathcal{R} = \mathcal{C}/\sim$, where $\mathcal{C}$ is the set of Cauchy Sequences of rational numbers, and where two sequences are equivalent, $(a_i) \sim (b_i)$, if for every positive rational number ϵ, there is $N \in \mathbb{N}$ so that*

$$|a_i - b_i| < \epsilon \qquad \text{whenever } i \geq N.$$

If $[(a_i)] \in \mathcal{R}$, define $[(a_i)] > 0$. Let $\mathcal{P} = \{[(a_i)] \in \mathcal{R} : [(a_i)] > 0\}$ denote the positive cone in $\mathcal{R}$. Show that if $[(a_i)], [(b_i)] \in \mathcal{P}$ then so is their product $[(a_i)][(b_i)] \in \mathcal{P}$.

$[(a_i)] > 0$ means that there is a rational number $\epsilon > 0$ and an $m \in \mathbb{N}$ such that

$$a_i > \epsilon \qquad \text{whenever } i \geq m.$$

Now suppose $[(a_i)] > 0$ and $[(b_i)] > 0$. There are rational number $\epsilon_1 > 0$, $\epsilon_2 > 0$ and $m_1, m_2 \in \mathbb{N}$ such that

$$a_i > \epsilon_1 \quad \text{whenever } i \geq m_1 \qquad \text{and} \qquad b_i > \epsilon_2 \quad \text{whenever } i \geq m_2.$$

Let m
$max\{m_1, m_2\}$. By multiplying, there is $\epsilon_1\epsilon_2 > 0$ and $m \in \mathbb{N}$ so that

$$a_i b_i > \epsilon_1 \epsilon_2 \qquad \text{whenever } i \geq m.$$

It follows that $[(a_i)][(b_i)] = [(a_i b_i)] > 0$, using the definition of multiplication in $\mathcal{R}$.

4. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) Statement *The middle thirds Cantor set C is uncountable.*

   True. The Cantor set may be realized as all ternary expansions involving only zeros and twos, namely,

   $$C = \left\{ \sum_{k=1}^{\infty} \frac{d_i}{3^i} : d_i \in \{0, 2\} \right\}$$

   Thus C is in one-to-one correspondence with the set of infinite strings of zeros and twos which is the Cartesian product $\prod_{k \in \mathbb{N}} \{0, 2\}$. But this is uncountable by Cantor's diagonal argument.

   (b) Statement. *Let X be a nonempty set. Define $d(x, y) = 1$ if $x = y$ and $d(x, y) = 0$ otherwise. Then (X, d) is a metric space.*

   False. A metric has to satisfy $d(x, x) = 0$ but here $d(x, x) = 1$.

   If the d were defined instead by $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ otherwise, then that is a metric, called the discrete metric.

   (c) Statement. *Let $B_\alpha \subset \mathbf{R}$ be open sets for each $\alpha \in \mathcal{I}$, where $\mathcal{I}$ is any index set. Then the intersection $\bigcap_{\alpha \in \mathcal{I}} B_\alpha$ is open.*

   False. Consider $B_n = (-\frac{1}{n}, \frac{1}{n})$ in the reals. Then $\bigcap_{n \in \mathbb{N}} B_n = \{0\}$ which is not open.

5. *Suppose that real numbers are partitioned into two nonempty subsets L and U such that each element of L is less that each element of U. Show that either L has a greatest element or U has a least element.*

Conpleteness of the reals gives the answer using "divide and conquer," the bisection procedure.

Since they are nonempty, one can pick $x_1 \in L$ and $y_1 \in U$ so $x_1 < y_1$. Supposing that $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ have been chosen we set $m_k = \frac{1}{2}x_k + y_k$. Then we let

$$\begin{aligned} x_{n+1} &= x_n, \quad y_{n+1} = m_n, \qquad \text{if } m_n \in U; \\ x_{n+1} &= m_n, \quad y_{n+1} = y_n, \qquad \text{if } m_n \in L; \end{aligned}$$

This gives $x_k \in L$, $y_k \in U$, $[x_{k+1}, y_{k+1}] \subset [x_k, y_k]$ and $|y_k - x_k| = 2^{1-n}|y_1 - x_1|$ for all k. Thus $\{x_n\}$ and $\{y_n\}$ are equivalent Cauchy sequences. By completeness, both sequences converge to $c \in \mathbf{R}$,

$$\lim_{k\to\infty} x_n = c = \lim_{k\to\infty} y_n.$$

It remains to argue that c is a maximum of L or a minimum of U. c is in one of the sets.

In case $c \in L$, then c is a maximum of L. Indeed, we have $\ell < y_n$ for every $\ell \in \mathcal{L}$, thus y_n is an upper bound of L. Since $y_n \to c$ then c is an upper bound of L. As $c \in L$, it is the maximum of L.

In case $c \in U$, then c is a minimum of U. Since $x_n \in L$ the x_n's are lower bounds of U. No smaller number than c can be a lower bound