Math 521 § 1. **Practice Problems for Final** Name:__________________
Treibergs Dec. 14, 1995

The following problems were taken from practie problems and final given Dec. 14, 1995.

---

1. Determine whether the following statements are true or false. If true, give a short argument. If false, give a counterexample.

   (a) Let (X, d) be a metric space. Then there is an $E \subset X$ such that E is neither open nor closed.

TRUE: ◯	FALSE: ◯

   (b) Let (X, d) be a metric space and $A \subset X$ be a subset and $x \in X$ be some point. Then there is $a \in X$ so that $d(x, a) = d(A, a)$.

TRUE: ◯	FALSE: ◯

   (c) In all metric spaces (X, d), there are always sets which are neither open nor closed.

TRUE: ◯	FALSE: ◯

   (d) Let $A \subset (X, d)$ be a subset of a metric space. Then the closure of the interior equals the closure $\overline{A^\circ} = \overline{A}$.

TRUE: ◯	FALSE: ◯

   (e) Let $A \subset (X, d)$ be a subset of a metric space. If A is open and dense then $A = X$.

TRUE: ◯	FALSE: ◯

   (f) Let $A \subset (X, d)$ be a subset of a metric space. If A is connected and $f : A \to Z$ is a continuous function with integer values, then f is constant.

TRUE: ◯	FALSE: ◯

   (g) Let (D, d) denote the disk $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ in the Euclidean plane $(\mathbb{R}^2, d)$. Then the completion is $(\overline{D}, d)$ the closed disk with the metric of the plane.

TRUE: ◯	FALSE: ◯

   (h) Let Q denote all open rectangles of $\mathbb{R}^2$. (*e.g.* sets of the form $(a, b) \times (c, d)$.) Then Q is an open base for the usual topology of $\mathbb{R}^2$. .

TRUE: ◯	FALSE: ◯

   (i) Suppose $f_n \in \mathcal{C}([0, 1])$ with sup-norm be a Cauchy sequence of nondecreasing functions. Then f_n converges to a continuous nondecreasing function.

TRUE: ◯	FALSE: ◯

   (j) Suppose A is a closed and bounded subset in a metric space. Then A is compact.

TRUE: ◯	FALSE: ◯

   (k) Suppose Suppose (X, d) is a compact metric space and $J \subset X$ is a closed subset. Then J is compact.

TRUE: ◯	FALSE: ◯

   (l) Suppose $K \subset U \subset \mathbb{R}^2$ be subsets of the plane such that K is compact and U is open. Then there is a polygonal curve (made of finitely many line segments joined end to end) curve in U which surrounds K.

TRUE: ◯	FALSE: ◯

(m) A function $h : [a, b] \to \mathbb{R}$ is called *k-Lipschitz* iff there is a constant $k < \infty$ so that for all $x, y \in [a, b]$, $|h(x)?h(y)|?k|x?y|$. Assume that $f_n \to g$ uniformly in $[a, b]$ and that for each n, $f_n(x)$ is k-Lipschitz with the same k. Show that g is Lipschitz. .

TRUE: ◯ FALSE: ◯

(n) Let $\{a_n\}$ be a sequence in a metric space (X, d) and let $V = \{a_n : n \in \mathbb{N}\}$ be the set of values. If x is a limit point of the sequence a_m then x is an cluster point (limit point) of V.

TRUE: ◯ FALSE: ◯

(o) The sequence $f_n(x) = \begin{cases} nx, & \text{if } nx < 1; \\ 1, & \text{if } nx \geq 1 \end{cases}$ converges in $\mathcal{C}([0, 1])$ to the discontinuous function $h(x) = \begin{cases} 0, & \text{if } x = 0; \\ 1, & \text{if } x > 0 \end{cases}$. .

TRUE: ◯ FALSE: ◯

(p) Suppose that $g : X \to R$ is a continuous function on a metric space. Then the sublevel set $S = \{x \in X : g(x) \leq 0\}$ is closed.

TRUE: ◯ FALSE: ◯

(q) Suppose A is a subset of a metric space (X, d) and let $x, y \in X$. Then $d(x, A) \leq d(x, y) + d(y, A)$.

TRUE: ◯ FALSE: ◯

(r) Suppose X is a metric space which is homeomorphic to $\mathbb{R}$ with its usual topology. Then X is complete.

TRUE: ◯ FALSE: ◯

(s) If X is a separable metric space, then X has a countable base for its topology.

TRUE: ◯ FALSE: ◯

2. Definition: Let E be a subset of a metric space (X, d). A point $x \in X$ is a *limit point* of E if and only if every neighborhood of x contains infinitely many distinct elements of E. Let $E?$ denot the set of all limit points of E. Show that $E?$ is closed. Does the set of limit points equal the closure $E? = \overline{E}$?

3. The diameter of a set E in a metric space (X, d) is defined to be

$$\operatorname{diam}(E) := \sup_{(x,y) \in E \times E} d(x, y)$$

Show that if E is compact, then there are two points $p, q \in E$ such that $d(p, q) = \operatorname{diam}(E)$.

4. Let $E \subset \mathbb{R}$ be the subset given by

$$E = \left\{ -1, -\frac{1}{2}, -\frac{1}{3}, \dots \right\} \cup \{x \in \mathbb{Q} : 1 < x < 2\} \cup [3, 4) \cup (4, 5)$$

(a) Describe the interior E°, the boundary ∂E, the closure $\overline{E}$, closure of the complement $\overline{E^c}$ and the cluster points $E?$.

(b) Is E totally bounded? Why? (take the usual metric of $\mathbb{R}$.)

5. Let (ℓ^∞, d) be the complete metric space of all bounded sequences $x = (a_i)$ of real numbers where for $x = (a_i)$, $y = (b_i)$ the distance is defined by

$$d(x, y) = \sup_{i \in \mathbb{N}} |a_i?b_i|.$$

Let $S_0 \in \ell^\infty$ be the subset consisting of those sequences such that a_i is nonzero for at most finitely many i. (For all $x \in S_0$ there is an $N \in \mathbb{N}$ so that $a_i = 0$ whenever $i \geq N$.) Show that the subspace (S, d) is incomplete. Let $\overline{S_0}$ denote the completion of (S_0, d). Find a subspace L isometric to $\overline{S_0}$ such that $S_0 \subset L \subset \ell^\infty$.

6. A real valued function f on a metric space (X, d) is called upper semicontinuous iff

$$(\forall x \in X)(\forall \epsilon > 0)(\exists \delta > 0)(\forall y \in X)[d(x, y) < \delta \implies f(y) < f(x) + \epsilon)]$$

   (a) Give an example of an upper semicontinuous function on $\mathbb{R}$ which is not continuous.

   (b) Suppose f is an upper semicontinuous function on the compact metric space (X, d). Show that f is bounded above.