Math 5410 § 1.	**Third Midterm Exam**	Name: Solutions
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1. *Find the value of the parameter a where a bifurcation occurs in the planar system and describe the nature of the bifurcation.*

$$\dot r = ar + r^3 + r^5$$
$$\dot\theta = 1 + r^2$$

A subcritical Hopf Bifurcation occurs at $a = 0$. Note that $\dot\theta$ is never zero so the flow circulates around the fixed point at the origin. The radial rest points are at radii that satisfy

$$0 = \dot r = r(a + r^2 + r^4)$$

There is one real root at $r_1 = 0$ and a second root if $a < 0$. Using the binomial expansion

$$r^2 = \frac{-1 \pm \sqrt{1-4a}}{2} \approx -\frac{1}{2} + \frac{1}{2}\left(1 - 2a - 2a^2 + \cdots\right) \approx -a$$

the other root is at $r_2 \approx \sqrt{-a}$. In that case $\dot r < 0$ if $r < r_2$ so the origin is stable and $\dot r > 0$ if $r > r_2$. Thus there is an unstable limit cycle at $r = r_2 \approx \sqrt{-a}$. As a increases to zero the unstable cycle collapses on the origin. For $a \ge 0$ there is only one root $r_1 = 0$ and $\dot r > 0$ for $r > 0$. Thus for $a \ge 0$ the origin is an unstable rest point. Since the cubic term is positive, this is a subcritical Hopf Bifurcation: an unstable limit cycle collapses on a stable origin, making it unstable at the bifurcation value.

2. *Determine whether the given equilibrium point for the given system is Asymptotically Stable, is Liapunov Stable but Not Asymptotically Stable, or is Not Stable. Give a brief explanation.*

(a) $r = 1,\ \theta = 0$ *for* $\begin{cases} \dot r = r(1-r^2) \\ \dot\theta = \sin^2\theta \end{cases}$

Not Stable. $\dot\theta > 0$ for $0 < \theta < \pi$ and for $\pi < \theta < 2\pi$. Thus on the unit circle where $\dot r = 0$, the flow is away from $(r,\theta) = (1,0)$ for $\theta > 0$. The θ phase line is

$$\longrightarrow 0 \longrightarrow \pi \longrightarrow 2\pi \longrightarrow$$

(b) $r = 1,\ \theta = \pi$ *for* $\begin{cases} \dot r = r(1-r^2) \\ \dot\theta = \sin\theta \end{cases}$

Asymptotically Stable. $\dot r > 0$ for $r < 1$ and $\dot r < 0$ for $r > 1$ so $r = 1$ is a stable rest point. Similarly $\dot\theta > 0$ for $0 < \theta < \pi$ and $\dot\theta < 0$ for $\pi < \theta < 2\pi$, thus $\theta = \pi$ is a sink. The linearization in the (r,θ) coordinates is $df_{(1,\pi)} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, which is a stability matrix. Linearization stability then says $(r,\theta) = (1,\pi)$ is asymptotically stable.

(c) $X = (0,0,0,0)$ *for* $\dot X = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} X$

Stable but Not Asymptotically Stable. This is a canonical matrix. The flow is an improper stable node in the first two variables and a center in the last two.

The eigenvalues are $-1, -1, i, -i$. Some eigenvalues have $\Re\lambda = 0$ so the origin is not asymptotically stable but the eigenvalues for which $\Re\lambda = 0$ have multiplicity one, so the origin is stable.

(d) $(x, y) = (0, 0)$ *for* $\begin{cases} \dot{x} = -y - x^3 \\ \dot{y} = x - y^3 \end{cases}$

Asymptotically Stable. Consider the Liapunov function $L(x, y) = x^2 + y^2$. It is positive definite and zero only at the rest point.

$$\dot{L} = \frac{d}{dt}\big(L(x(t), y(t)\big) = 2x\dot{x} + 2y\dot{y} = 2x(-y - x^3) + 2y(x - y^3) = -2x^4 - 2y^4 < 0$$

for $(x, y) \neq (0, 0)$. Hence L satisfies the strong condition for decrease and the rest point is asymptotically stable.

3. *Let the $n \times n$ matrix $A(t)$ and the vector $b(t) \in \mathbf{R}^n$ be continuous function for $0 \le t \le T$. Show that the solution of the initial value problem is unique on $[0, T]$.*

$$\begin{aligned} \dot{x} &= A(t)x + b(t) \\ x(0) &= c \end{aligned}$$

We know that for linear equations with continuos A and b, the solution exists on $[0, T]$. Of several proofs, let's use the Gronwall inequality method. Since $A(t)$ is continuous, its operator norm is continuous and has a finite maximum over the compact interval $[0, T]$

$$L = \sup_{0 \le t \le T} \|A(t)\|.$$

To prove uniqueness, suppose that we have two solutions $x(t)$ and $y(t)$. They satisfy the integral equation for $0 \le t \le T$

$$\begin{aligned} x(t) &= x(0) + \int_0^t \big\{A(s)x(s) + b(s)\big\}\, ds \\ y(t) &= y(0) + \int_0^t \big\{A(s)y(s) + b(s)\big\}\, ds \end{aligned}$$

Subtracting,

$$x(t) - y(t) = x(0) - y(0) + \int_0^t A(s)\big(x(s) - y(s)\big)\, ds$$

Taking norms, estimating the integral and using the operator norm,

$$\begin{aligned} |x(t) - y(t)| &\le |x(0) - y(0)| + \int_0^t \big|A(s)\big(x(s) - y(s)\big)\big|\, ds \\ &\le |x(0) - y(0)| + \int_0^t \|A(s)\|\, \big|x(s) - y(s)\big|\, ds \\ &\le |x(0) - y(0)| + \int_0^t L\, \big|x(s) - y(s)\big|\, ds \end{aligned}$$

By Gronwall's Inequality, for all $0 \le t \le T$,

$$|x(t) - y(t)| \le |x(0) - y(0)|e^{Mt} \le |x(0) - y(0)|e^{MT}$$

But since $x(0) - y(0) = c - c = 0$, this says $x(t) = y(t)$ for all $0 \le t \le T$. Thus all solutions are equal so are unique.

As a word of warning, the system cannot be integrated as in $\mathbf{R}^1$. The problem is that the solution of $\dot{X} = A(t)X$, $X(0) = I$ is not given by

$$\exp\left(\int_0^t A(s)\,ds\right)$$

unless $A(u)A(v) = A(u)A(v)$ for all $0 \le u, v \le T$.

4. *Let $\varphi(t, c)$ be the solution of the initial value problem.*

$$\begin{aligned} \dot{x} &= (x - \sin t)x + \cos t \\ x(0) &= c. \end{aligned}$$

Find $\left.\frac{\partial}{\partial c}\right|_{c=0} \varphi(\pi, c)$. *[Hint: $\varphi(t, 0) = \sin t$.]*

The derivative with respect to initial values at c

$$\Phi(t, c) = \frac{\partial \varphi}{\partial c}(t, c)$$

satisfies the variational equation along the solution $\varphi(t, c)$,

$$\begin{aligned} \frac{d\Phi}{dt}(t, c) &= \frac{\partial f}{\partial x}(t, \varphi(t, c))\, \Phi(t, c) \\ \Phi(0, c) &= 1. \end{aligned}$$

In this case,

$$f(t, x) = (x - \sin t)x + \cos t$$

so

$$\frac{\partial f}{\partial x}(t, x) = 2x - \sin t$$

If $c = 0$ so $\phi(t, 0) = \sin t$ we have

$$\frac{\partial f}{\partial x}(t, \varphi(t, 0)) = 2\sin t - \sin t = \sin t.$$

Thus the variational equation is

$$\begin{aligned} \frac{d\Phi}{dt}(t, 0) &= \sin t\, \Phi(t, 0) \\ \Phi(0, 0) &= 1. \end{aligned}$$

This equation separates

$$\frac{d\Phi}{\Phi} = \sin t\, dt$$

so

$$\log \Phi(t, 0) = \log \Phi(t, 0) - \log \Phi(0, 0) = 1 - \cos t$$

or

$$\Phi(t, 0) = e^{1 - \cos t}.$$

Thus $\Phi(\pi, 0) = e^2$.

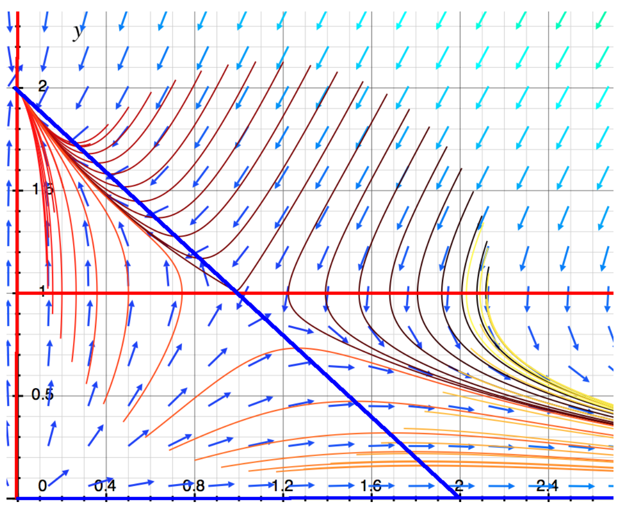
5. *You may assume that the system is defined only for $x, y \geq 0$. The equilibrium points are $(0,0)$, $(0,2)$ and $(1,1)$.*

$$\dot{x} = x(1-y)$$
$$\dot{y} = y(2-x-y)$$

Draw the x and y nullclines and use this information to sketch the global phase portrait of this nonlinear system. At the interior equilibrium point $(1,1)$, give a detailed description of the behavior of the linearized system.

To confirm the given information, the rest points occur at $0 = \dot{x} = x(1-y)$ so either $x = 0$ or $y = 1$ and at $0 = \dot{y} = y(2-x-y)$ so either $y = 0$ or $x + y = 2$. There are three possibilities, either both x and y are zero $(0,0)$, $y = 1$ so $2 = x + 1$ whis gives the point $(1,1)$ or $x = 0$ and $2 = 0 + y$ which gives the point $(0,2)$. We can't have both $y = 1$ and $y = 0$.

The $0 = \dot{x} = x(1-y)$ nullclines are $x = 0$ or $y = 1$. The flow is vertical on these two lines. On $x = 0$, $\dot{y} = y(2-y)$ is upward for $0 < y < 2$ and downward for $2 < y$. On $y = 1$ $\dot{y} = 1-x$ is upward for $0 < x < 1$ and downward for $1 < x$. The $0 = \dot{y} = y(2-x-y)$ nullclines are $y = 0$ and $2 = x + y$ where the flow is horizontal. On the $y = 0$ nullcline, $\dot{x} = x$ is always to the right. On the $2 = x + y$ nullcline for $x > 0$ we have $\dot{x} = x(1-[2-x]) = x(x-1)$ so the flow is to the left for $0 < x < 1$ and to the right for $1 < x \leq 2$.



$\dot{x} = 0$ nullcline (red), $\dot{y} = 0$ nullcline (blue), vector field and trajectories.

The Jacobian is

$$J(x,y) = \begin{pmatrix} 1-y & -x \\ -y & 2-x-2y \end{pmatrix}$$

At the interior rest point,

$$J(1,1) = \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix}$$

whose determinant is $\Delta = -1$, so the rest point $(1,1)$ is a saddle. The characteristic polynomial is

$$\begin{vmatrix} -\lambda & -1 \\ -1 & -1-\lambda \end{vmatrix} = \lambda^2 + \lambda - 1$$

whose roots are $\lambda_\pm = \frac{1}{2}(-1 \pm \sqrt{5})$. The eigenvectors satisfy

$$0 = \begin{pmatrix} \frac{1}{2}(1 \mp \sqrt{5}) & -1 \\ -1 & \frac{1}{2}(-1 \mp \sqrt{5}) \end{pmatrix} \begin{pmatrix} 2 \\ 1 \mp \sqrt{5} \end{pmatrix}$$

Thus the directions of the stable $\mathcal{W}^s$ and the unstable $\mathcal{W}^u$ lines at $(1,1)$ corresponding to the negative and positive eigenvalues λ_- and λ_+, respectively, are

$$w_- = \begin{pmatrix} 2 \\ 1+\sqrt{5} \end{pmatrix}, \qquad w_+ = \begin{pmatrix} 2 \\ 1-\sqrt{5} \end{pmatrix}.$$

We complete the discussion of the Jacobians at the other rest points, although this was not required for the answer.

At the rest point at the origin,

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$ with eigenvectors along the x and y axes, resp. The origin is an unstable node.

Thus the slow and fast outgoing directions at the origin are, resp., $w_1 = \binom{1}{0}$, $w_2 = \binom{0}{1}$.

At the remaining boundary rest point,

$$J(0,2) = \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}$$

has eigenvalues $\lambda = -1, -1$. This is a stable improper node with a single eigenvector $w = \binom{0}{1}$. The flow along the y axis approaches $(0,2)$. The incoming flow from $x > 0$ approaches $(0,2)$ but hooks at the last instant and approaches tangent to w.