Math 5410 § 1. **First Midterm Exam** Name: Solutions
Treibergs Sept. 21, 2022

1. *Consider the differential equation. Find the general solution. Find the Poincaré map for 2π periodic solutions. Is there a 2π-periodic solution? is it unique? Why?*

$$x' = \sin(t)(1 + x).$$

The general solution may be found by separating variables. Suppose $x(0) = x_0$. Then separating and integrating we get

$$\ln\left(\frac{x+1}{x_0+1}\right) = \ln(x+1) - \ln(x_0+1) = \int_{x_0}^{x} \frac{dx}{x+1} = \int_0^t \sin s\, ds = 1 - \cos t$$

so

$$\frac{x(t)+1}{x_0+1} = e^{1-\cos t}$$

which implies

$$x(t) = (x_0+1)e^{1-\cos t} - 1.$$

Checking, $x(0) = (x_0 + 1) - 1 = x_0$. The Poincaré map is

$$p(x_0) = x(2\pi) = (x_0+1)e^{1-\cos 2\pi} - 1 = (x_0+1)\cdot 1 - 1 = x_0.$$

It follows that FOR EVERY CHOICE OF $x_0 \in \mathbf{R}$ we have $x_0 = p(x_0)$. Thus every trajectory is periodic so the periodic trajectories are not unique.

2. *Consider the system. Determine the canonical form for this equation. Find the matrix P so that $Y = PX$ puts (1) in canonical form. Check that your matrix works.*

$$X' = \begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix} X. \qquad (1)$$

From the characteristic polynomial

$$0 = \det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (5-\lambda)(3-\lambda) + 1 = \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2.$$

we see that $\lambda = 4, 4$ with algebraic multiplicity two. We can find an eigenvector for $\lambda = 4$.

$$0 = (A - \lambda I)V = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The matrix $A - \lambda I$ has rank one, so there are no other eigenvalues. It follows that the canonical form must be

$$J = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}.$$

A matrix that brings A to canonical form may be constructer from the eigenvector and its corresponding cyclic vector. However, we can also use the recipe from the text. Choose any independent vector

$$W = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Now we express in the basis $\{V, W\}$,

$$\begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} = AW = \mu V + \nu W = (-1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then the matrix

$$P = \left(V, \frac{1}{\mu} W \right) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

will do the job. We see this by checking

$$AP = \begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix} = PJ.$$

3. *Find the flows ϕ_t^X and ϕ_t^Y. Find an explicit topological congugacy between the flows. Check that your conjugacy works.*

$$X' = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} X, \qquad Y' = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} Y.$$

Both equations decouple. Thus, solving componentwise we find the flows

$$\phi_t^X \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 e^{3t} \\ x_2 e^{t} \end{pmatrix}, \qquad \phi_t^Y \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 e^{2t} \\ y_2 e^{5t} \end{pmatrix}$$

The desired topological conjugacy is defined componentwise

$$H \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} h_1(x_1) \\ h_2(x_2) \end{pmatrix}$$

where each of the $h_i : \mathbf{R} \to \mathbf{R}$ are homeomorphisms (continuous map with continuous inverse) so their product $H : \mathbf{R}^2 \to \mathbf{R}^2$ is a homeomorphism. The h_i are appropriate powers to map the component flows from x to y. Thus we put

$$h_1(x_1) = \operatorname{sgn}(x_1)|x_1|^{2/3},$$
$$h_2(x_2) = \operatorname{sgn}(x_2)|x_2|^{5}.$$

To see that this gives a flow conjugacy, for any t and $(x_1, x_2) \in \mathbf{R}^2$ we have

$$H \circ \phi_t^X \begin{pmatrix} x_1 \\ \\ x_2 \end{pmatrix} = H \begin{pmatrix} x_1 e^{3t} \\ \\ x_2 e^t \end{pmatrix} = \begin{pmatrix} \operatorname{sgn}(x_1 e^{3t})|x_1 e^{3t}|^{2/3} \\ \\ \operatorname{sgn}(x_2 e^t)|x_2 e^t|^5 \end{pmatrix} = \begin{pmatrix} \operatorname{sgn}(x_1)|x_1|^{2/3} e^{2t} \\ \\ \operatorname{sgn}(x_2)|x_2|^5 e^{5t} \end{pmatrix}$$
$$= \phi_t^Y \begin{pmatrix} \operatorname{sgn}(x_1)|x_1|^{2/3} \\ \\ \operatorname{sgn}(x_2)|x_2|^5 \end{pmatrix} = \phi_t^Y \circ H \begin{pmatrix} x_1 \\ \\ x_2 \end{pmatrix}.$$

4. *Consider the system. Sketch the regions in the ab-plane where this system has different types of canonical forms. In the interior of each region, sketch a small phase plane indicating how the flow looks.*

$$X' = \begin{pmatrix} a & b \\ \\ 1 & 0 \end{pmatrix} X.$$

The characteristic polynomial

$$0 = \det(A - \lambda I) = \begin{vmatrix} a - \lambda & b \\ \\ 1 & -\lambda \end{vmatrix} = (a - \lambda)(-\lambda) - b = \lambda^2 - a\lambda - b.$$

The eigenvalues from the quadratic formula are

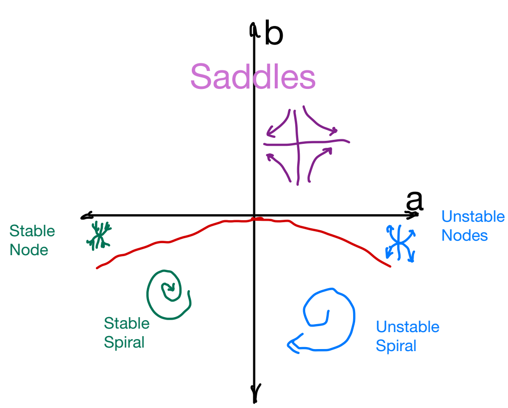
$$\lambda_\pm = \frac{a \pm \sqrt{a^2 + 4b}}{2}$$

Thus they are complex, conjugate if $4b < -a^2$, so the solutions are spirals. They have opposite signs if $b > 0$ because then $\sqrt{a^2 + 4b} > |a|$, which is the case of a saddle. Finally if $-a^2 < 4b < 0$ then $\lambda_\pm$ are unequal but have the same signs, which is the case of proper nodes.

On the line $b = 0$ one eigenvalue is zero, which is a degenerate cases of combs, shear and whole plane of rest points. On the line of $a^2 + 4b = 0$ we have repeated eigenvalue, which gives improper nodes for $b < 0$.

Finally, if $b < 0$ then both eigenvalues have $\Re\mathrm{e}(\lambda_i) < 0$ (sink) if $a < 0$ and $\Re\mathrm{e}(\lambda_i) > 0$ (source) if $a > 0$.

The different main cases are summarized by in the a, b plane



5. *Consider the family of differential equations depending on the parameter a Find the bifurcation points. Sketch the bifurcation diagram for this family of equations. Identify the rest points on the bifurcation diagram as sources, sinks or neither. Sketch the phase lines for values of a above and below the bifurcation values.*

$$x' = (x-2)(x+a)$$

Let $f(x,a) = (2-x)(x+a)$. The bifurcation curve is the locus of $f(x,a) = 0$, in other words, the rest points are at

$$x = 2 \qquad \text{or} \qquad x = -a.$$

The bifurcation point is the intersection at $(a,x) = (-2,2)$ where the lines intersect. Since $f(x,a) > 0$ between $x = 0$ and $x = a$ and $f(x,0) < 0$ for $x < \min(2,a)$ and for $x > \max(2,a)$, the flow is toward the positive between the rest points and toward the negative outside the rest points. Hence, away from the bifurcation points, the lower rest point is unstable and the upper one is stable. This is a trans-critical bifurcation. The phase lines just above and just below the bifurcation value $a = 2$ are shown.