Midterm 2 (5440)

Name and Unid: ________________________________

Carefully Read The Instructions:

Instructions: This exam will last 50 minutes and consists of 5 exercises and one bonus exercise. Provide solutions to the exercises in the space provided or if you need extra space to work, there are two pages at the end of the test, but please idicate the exercise. All solutions MUST be sufficiently justified to receive all the credit. Illegible answers will receive deductions. Calculators, books and notes are not allowed.

Advice: If you get stuck on a exercise or a question don't panic! Move on and come back to it later.

Exercise 1(Fourier transform of a centered Gaussian function).

We recall that the Fourier transform $\hat{u}$ of a function u is given by:

$$\hat{u}(\xi) = \int_{\mathbb{R}} u(x)\, e^{-i\xi x} \mathrm{d}x, \ \forall \xi \in \mathbb{R}$$

1) We denote by g the Gaussian function:

$$g(x) = e^{-ax^2}, \quad \forall x \in \mathbb{R}.$$

with a a fixed positive number and by $\hat{g}(\xi)$ its Fourier transform define by

$$\hat{g}(\xi) = \int_{\mathbb{R}} e^{-ax^2}\ e^{-i\xi x} \mathrm{d}x, \ \forall \xi \in \mathbb{R}.$$

By interchanging the derivation in ξ and the integral in x (no justification is asked here) in the expression

$$\frac{\mathrm{d}\hat{g}}{\mathrm{d}\xi}(\xi) = \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\int_{\mathbb{R}} e^{-ax^2}\ e^{-i\xi x} \mathrm{d}x\right),$$

give an integral expression of $\frac{\mathrm{d}\hat{g}}{\mathrm{d}\xi}(\xi)$ (which does not include any derivative symbol).

2) By integrating by part the expression found in the question 1), prove that $\hat{g}(\xi)$ satisfy the following ODE.

$$\frac{\mathrm{d}\hat{g}}{\mathrm{d}\xi}(\xi) = -\frac{\xi}{2a}\,\hat{g}(\xi). \tag{1}$$

3) We recall that

$$\int_{-\infty}^{\infty} e^{-x^2} \mathrm{d}x = \sqrt{\pi},$$

using this formula compute the value of $\hat{g}(0)$.

4) By solving the first order differential equation (1) with the initial condition $\hat{g}(0)$, express $\hat{g}(\xi)$.

5) (Bonus question) Suppose that a is a large positive real number. Give a sketch of the functions $g(x)$ and $\hat{g}(\xi)$.

6) (Bonus question) Prove the formula

$$\int_{-\infty}^{\infty} e^{-x^2} \mathrm{d}x = \sqrt{\pi},$$

used in the question 3.
(*Compute first* $\int_0^\infty e^{-(x^2+y^2)}\mathrm{d}x\,\mathrm{d}y$ *by expressing this integral in polar coordinates, then deduces the value of* $\int_0^\infty e^{-x^2}\mathrm{d}x$ *and finally the value of* $\int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x$.)

<u>Exercice 2</u> (Separation of variables of the heat equation with Dirichlet boundary conditions).

We consider the following heat equation:

$$\left| \begin{array}{l} \dfrac{\partial u}{\partial t}(x,t) = k\dfrac{\partial^2 u}{\partial x^2}(x,t), \qquad \text{for } 0 < x < L \text{ and } t > 0, \\ \\ u(0,t) = 0 \ \text{ and } \ u(l,t) = 0, \end{array} \right. \tag{2}$$

1) Apply the separations of variables method by looking for non zero solutions

$$u(x,t) = X(x)\,T(t)$$

of this problem. Find one boundary value problem for X (ordinary differential equation+boundary conditions at $x = 0$ and $x = l$) and one ODE for T.

2) Prove that the boundary value problem satisfied by X admits only positive eigenvalues.

3) Find the eigenvalues λ and the eigenfunctions associated to the boundary value problem satisfied by X.

4) Solve the time ODE for λ an eigenvalue.

5) By "using the superposition principle" (no justification is asked here) express the general form of the solution.

6) (bonus equation) We suppose that the heat equation () admits as initial condition

$$u(x,0) = 1 \qquad \text{for } 0 \le x \le L.$$

By computing the unknown coefficients which appears in the general form of the solution of the question 5, precise the expression of the solution with this particular initial condition.

We recall the formula of a sine Fourier series, if

$$\phi(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right),$$

then

$$B_n = \frac{2}{L}\int_0^L \phi(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \quad \text{for all positive integer } n.$$

Exercise 3: (Stability of the heat equation on the whole real line).

We denote by $\|\cdot\|_{\infty,1}$ the uniform norm on the continuous bounded function on $\mathbb{R}$ defined by:

$$\|\phi\|_{\infty,1} = \max_{x\in\mathbb{R}} |\phi(x)|,$$

(To be rigorous, I shall replace here the word max by sup because the maximum of ϕ could be reached at $x = \pm\infty$).

We denote by $\|\cdot\|_{\infty,2}$ the uniform norm on the continuous bounded function on $\mathbb{R}\times\mathbb{R}^+$ defined by:

$$\|u\|_{\infty,2} = \max_{x\in\mathbb{R},\, t\in\mathbb{R}^+} |u(x,t)|.$$

We consider the following heat equation on the whole real line

$$(P)\left|\begin{array}{l} \dfrac{\partial u}{\partial t}(x,t) = k\dfrac{\partial^2 u}{\partial x^2}(x,t) \ \text{ for } t>0 \text{ and } x\in\mathbb{R}, \\[2ex] u(x,0) = \phi(x),\ \forall x\in\mathbb{R}. \end{array}\right.$$

We recall that the solution of the problem (P) is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi k\,t}} \int_{\mathbb{R}} \exp\left(-\frac{(x-y)^2}{4\,kt}\right)\phi(y)\,\mathrm{d}y,\ \ \forall x\in\mathbb{R} \text{ and } \forall t>0. \tag{3}$$

1) Prove by a change a variable that the formula (3) could be rewritten as

$$u(x,t) = \frac{1}{\sqrt{4\pi}} \int_{\mathbb{R}} \exp\left(-\frac{p^2}{4}\right)\,\phi(x - p\sqrt{kt})\,\mathrm{d}p\ . \tag{4}$$

2) We denote by u_1 and u_2 the solutions of the problem (P) associated to the initial conditions ϕ_1 and ϕ_2. We call u and ϕ the functions $u = u_2 - u_1$ and $\phi = \phi_2 - \phi_1$. Which heat equation and initial condition is satisfied by the function u?

3) Use the formula

$$\int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x = \sqrt{\pi},$$

to compute the integral:

$$\int_{\infty}^{\infty} \exp\left(-\frac{p^2}{4}\right)\mathrm{d}p. \tag{5}$$

4) By using the formula (4) and the value of the integral (5), prove that:

$$\|u\|_{\infty,2} \leq \|\phi\|_{\infty,1} \,. \tag{6}$$

5) By applying the ϵ, δ definition of the stability on the formula (6), deduce the stability of the problem (P).