Math 5440 § 1. **Second Midterm Exam** Name: Solutions
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1. *Consider the initial-boundary value problem on an interval with $k > 0$ constant.*

$$\begin{aligned} &\text{(PDE)} && u_t = ku_{xx}, && \text{for } 0 < x < \frac{\pi}{3} \text{ and } 0 < t; \\ &\text{(BC)} && u(0,t) = 0, && \\ & && u(\frac{\pi}{3}, t) = 0, && \text{for } 0 < t; \\ &\text{(IC)} && u(x,0) = \varphi(x), && \text{for } 0 < x < \frac{\pi}{3}. \end{aligned}$$

Separate variables and deduce an eigenvalue problem for the x-part of the solution. Determine the eigenvalues and eigenfunctions. (You may assume eigenvalues are positive.) Solve the t-part of the solution and find the general solution as a series.

Assuming $u(t,x) = T(t)X(x)$, the separation of variables leads to

$$T'(t)X(x) = kT(t)X''(x)$$

so

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda$$

for some constant λ. The BC's imply this eigenvalue problem for X has Dirichlet conditions. The eigenvalue problem becomes

$$X'' + \lambda X = 0, \qquad X(0) = 0, \qquad X(\frac{\pi}{3}) = 0$$

We are told λ is positive. But this is easy to see because these are symmetric BC's so the eigenvalue is real, nonnegative. If $\lambda = 0$ then $X'' = 0$ so $X = A + Bx$. The boundary conditions give $0 = X(0) = A$ and $0 = X(\frac{\pi}{3}) = 0 + B\frac{\pi}{3}$ so $B = 0$ also, the trivial solution. So $\lambda = 0$ is ruled out leaving $\lambda > 0$.

Putting $\lambda = \beta^2$ where $\beta > 0$ gives solutions

$$X(x) = A\cos\beta x + B\sin\beta x$$

Boundary conditions $0 = X(0) = A$ and $0 = X(\frac{\pi}{3}) = 0 + B\frac{\pi}{3}$ imply $B = 0$ or

$$\beta_n = 3n, \qquad n = 1, 2, 3, \ldots.$$

The eigenfunctions are thus

$$X_n(x) = \sin 3nx, \qquad n = 1, 2, 3, \ldots.$$

The corresponding time equation is

$$T_n' + k\lambda_n T_n = 0, \qquad n = 1, 2, 3, \ldots$$

whose general solution is

$$T_n(t) = A_n e^{-9kn^2 t}$$

where A_n is constant. The general solution is thus

$$u(t,x) = \sum_{n=1}^{\infty} A_n e^{-9kn^2 t} \sin 3nx.$$

The coefficients satisfy

$$\varphi(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin 3nx$$

so A_n is the sine series coefficient, given by the formula

$$A_n = \frac{2}{\ell}\int_0^{\ell} \varphi(x) \sin\left(\frac{n\pi x}{\ell}\right)\, dx = \frac{6}{\pi}\int_0^{\frac{\pi}{3}} \varphi(x) \sin 3nx\, dx.$$

2. *Consider the initial-boundary value problem.*

$$\begin{array}{lrl} \text{(PDE)} & u_{tt} = c^2 u_{xx}, & \text{for } 0 < x < \pi \text{ and } 0 < t; \\ \text{(BC)} & u_x(0,t) = 0, & \\ & u(\pi,t) = 0, & \text{for } 0 < t; \\ \text{(IC)} & u(x,0) = 0, & \\ & u_t(x,0) = \cos \tfrac{5}{2}x, & \text{for } 0 < x < \pi. \end{array}$$

All solutions of the eigenvalue problem

$$X'' + \lambda X = 0, \qquad X'(0) = 0, \qquad X(\pi) = 0$$

are given by

$$X_n(x) = \cos\left(n + \frac{1}{2}\right) x, \qquad n = 0, 1, 2, 3, \ldots.$$

Find the eigenvalues. Find the general solution as a series. Find the particular solution.

Inserting the eigenfunction X_n into the PDE

$$-\left(n+\frac{1}{2}\right)^2 \cos\left(n+\frac{1}{2}\right) x + \lambda_n \cos\left(n+\frac{1}{2}\right) x$$

implies

$$\lambda_n = \left(n+\frac{1}{2}\right)^2, \qquad n = 0, 1, 2, 3, \ldots.$$

Assuming $u(t,x) = T(t)X(x)$, the separation of variables leads to

$$\frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda$$

for some constant given as positive $\lambda = \beta^2$ where $\beta > 0$. The BC's imply this eigenvalue problem for X has mixed BC Neumann on the left and Dirichlet on the right BC's. You were given the eigenfunctions $X_n(x)$. The corresponding time equation is

$$T_n'' + c^2 \lambda_n T_n = 0, \qquad n = 0, 1, 2, 3, \ldots$$

whose general solution is

$$T_n(t) = A_n \cos\left(n+\frac{1}{2}\right) ct + B_n \sin\left(n+\frac{1}{2}\right) ct$$

where A_n and B_N are constant. The general solution is thus

$$u(t,x) = \sum_{n=1}^{\infty} \left\{ A_n \cos\left(n+\frac{1}{2}\right) ct + B_n \sin\left(n+\frac{1}{2}\right) ct \right\} \cos\left(n+\frac{1}{2}\right) x$$

We have

$$0 = u(x,0) = \sum_{n=1}^{\infty} A_n \cos\left(n + \frac{1}{2}\right) x$$

so $A_n = 0$ for all n. Also

$$\cos\frac{5}{2}x = u_t(x,0) = \sum_{n=1}^{\infty} B_n c\left(n + \frac{1}{2}\right)\cos\left(n + \frac{1}{2}\right) x$$

so

$$B_2 = \frac{2}{5c}$$

and the rest of the $B_n = 0$. Thus the particular solution is

$$u(t,x) = \frac{2}{5c}\sin\left(\frac{5}{2}ct\right)\cos\left(\frac{5}{2}x\right).$$

3. *Consider the functions* $f_n(x) = x^n - x^{n+1}$. *The infinte sum* $S(x) = \sum_{n=1}^{\infty} f_n$ *is known to converges pointwise on the interval* $[0,1]$.

   (a) *Find* $S(x)$ *for* $0 \le x \le 1$. *[Be careful!]*
   The sum telescopes so

   $$S_n = (x^1 - x^2) + (x^2 + x^3) + \cdots (x^N - x^{N+1}) = x - x^{N+1}$$

   If $x = 1$ then $S_N(x) = 0$ for all N. On the other hand, for $0 \le x < 1$, $S_N(x) = x - x^{N+1} \to x$ as $N \to \infty$. Summarizing, for $0 \le x \le 1$,

   $$S(x) = \lim_{N\to\infty} S_N(x) = \begin{cases} x, & \text{if } 0 \le x < 1; \\ 0, & \text{if } x = 1. \end{cases}$$

   (b) *Define:* $\sum_{n=1}^{\infty} f_n$ *converges uniformly.*
   Does the series converge uniformly on $[0,1]$ *with these* f_n*? Why?*
   The series $\sum_{n=1}^{\infty} f_n$ is said to converge uniformly on $[0,1]$ if

   $$\lim_{N\to\infty} \sup_{0\le x\le 1} |S_N(x) - S(x)| = 0.$$

   Here the sum does not converge uniformly on $[0,1]$. Even though $S_N(1) - S(1) = 0$, this is because for each N,

   $$\sup_{0\le x\le 1} |S_N(x) - S(x)| = \sup_{0\le x<1} \left|x - x^{N+1} - x\right| = 1$$

   which does not converge to zero.

   (c) Define: $\sum_{n=1}^{\infty} f_n$ converges *in the* $\mathcal{L}^2$*-sense.*
   Does the series converge in the $\mathcal{L}^2$*-sense on* $[0,1]$ *with these* f_n*? Why?*

The series $\sum_{n=1}^{\infty} f_n$ is said to converge in the $\mathcal{L}^2$-sense on $[0,1]$ if

$$\lim_{N\to\infty} \int_0^1 |S_N(x) - S(x)|^2 \, dx = 0.$$

Here the sum does converge in the $\mathcal{L}^2$-sense on $[0,1]$. We can omit one point $|S_N(1) - S(1)|^2 = 0$ of the integrand and not change the integral. For each N,

$$\int_0^1 |S_N(x) - S(x)|^2 \, dx = \int_0^1 \left|x - x^{N+1} - x\right|^2 \, dx = \left[\frac{x^{2N+3}}{2N+3}\right]_0^1 = \frac{1}{2N+3}$$

which converges to zero as $N \to \infty$.

4. *For positive constants A, B, T let*

$$\varphi(x) = \begin{cases} 1, & \text{if } 0 \le x \le A; \\ 0, & \text{otherwise.} \end{cases} \qquad f(t,x) = \begin{cases} 1, & \text{if } 0 \le x \le B \text{ and } 0 \le t \le T; \\ 0, & \text{otherwise.} \end{cases}$$

Consider the initial-boundary value problem on the half-line, where $k > 0$ is constant.

(PDE)	$u_t - ku_{xx} = f(t,x),$	for $0 < x < \infty$ and $0 < t$;
(BC)	$u_x(0,t) = 0,$	for $0 < t$;
(IC)	$u(x,0) = \varphi(x),$	for $0 < x < \infty$.

Solve the problem. You may write your solution as an integral. Show that for all $x \ge 0$, the solution $u(x,t) \to 0$ as $t \to \infty$.

The Neumann condition at $x = 0$ tells us to find the even extension to all of $-\infty < x < \infty$ and then do Duhamel's formula using the extended data to express the solution on the halfline.

$$\varphi_{\text{ev}}(x) = \begin{cases} 1, & \text{if } -A \le x \le A; \\ 0, & \text{otherwise.} \end{cases} \qquad f_{\text{ev}}(t,x) = \begin{cases} 1, & \text{if } -B \le x \le B \text{ and } 0 \le t \le T; \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$u(t,x) = \int_{-\infty}^{\infty} S(t, x-y)\varphi_{\text{ev}}(y)\, dy + \int_0^t \int_{-\infty}^{\infty} S(t-s, x-y) f_{\text{ev}}(s,y)\, dy\, ds +$$

where

$$S(t,z) = \frac{1}{\sqrt{4k\pi t}} \exp\left(-\frac{z^2}{4kt}\right).$$

So, if $t > T$ then inserting φ_{ev} and F_{ev},

$$u(t,x) = \int_{-A}^{A} \frac{1}{\sqrt{4k\pi t}} \exp\left(-\frac{(x-y)^2}{4kt}\right) dy + \int_0^T \int_{-B}^{B} \frac{1}{\sqrt{4k\pi(t-s)}} \exp\left(-\frac{(x-y)^2}{4k(t-s)}\right) dy\, ds$$

The integrands are everywhere positive so $u(t,x) \ge 0$. Observing that for $0 \le s \le T < t$ we have

$$\exp\left(-\frac{(x-y)^2}{4kt}\right) \le 1, \qquad \exp\left(-\frac{(x-y)^2}{4k(t-s)}\right) \le 1, \qquad \frac{1}{\sqrt{4k\pi(t-s)}} \le \frac{1}{\sqrt{4k\pi(t-T)}}$$

It follows that for $t > T$,

$$0 \le u(t,x) \le \frac{2A}{\sqrt{4k\pi t}} + \frac{2BT}{\sqrt{4k\pi(t-T)}} \to 0, \qquad \text{as } t \to \infty.$$

5. *Consider the eigenvalue problem on* $1 \le x \le 2$

$$X'' + \lambda X = 0, \qquad X'(1) = X(1), \qquad X'(2) = -X(2).$$

Show that the eigenvalues are real and positive.

Let X be a possibly complex eigenfunction with possibly complex eigenvalue λ. Multiplying by the complex conjugate, integrating by parts and using the boundary condition yields

$$\begin{aligned}
\lambda \int_1^2 X\bar{X}\,dx &= -\int_1^2 X''\bar{X}\,dx \\
&= \int_1^2 X'\bar{X}'\,dx - \Big[X'\bar{X}\Big]_1^2 \\
&= \int_1^2 X'\bar{X}'\,dx - X'(2)\bar{X}(2) + X'(1)\bar{X}(1) \\
&= \int_1^2 X'\bar{X}'\,dx + X(2)\bar{X}(2) + X(1)\bar{X}(1) \\
\lambda \int_1^2 |X|^2\,dx &= \int_1^2 |X'|^2\,dx + |X(2)|^2 + |X(1)|^2
\end{aligned}$$

Since X is nontrivial, the integral of $|X|^2$ is positive. Similarly, the terms on the right are nonnegative, hence λ is real, nonnegative.

If $\lambda = 0$ then $X'' = 0$ so $X = A + Bx$ and $X' = B$. The boundary conditions give

$$\begin{aligned}
B &= X'(1) = A + B \\
B &= X'(2) = A + 2B
\end{aligned}$$

so $A = B = 0$, the trivial solution. So $\lambda = 0$ is ruled out leaving $\lambda > 0$.