Math 5440 § 1. **First Midterm Exam** Name: Practice Problems
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1. *Find an equation whose solutions are graphs over the plane which are rotationally symmetric about the z-axis.*

   We seek a PDE whose general solution is
   $$u(x,y) = f(x^2+y^2)$$
   We shall differentiate and eliminate f. To simplify notation, put $t = x^2+y^2$. Then
   $$u_x = f'(t)\,2x, \qquad u_y = f'(t)\,2y$$
   Equating
   $$\frac{u_x}{x} = 2f'(t) = \frac{u_y}{y}$$
   we see that the eliminating f give the desired PDE
   $$y\,u_x - x\,u_y = 0.$$
   Checking, we see that the characteristics satisfy the ODE's
   $$\dot{x}(\tau) = y; \qquad \dot{y}(\tau) = -x.$$
   These imply
   $$\frac{dy}{dx} = -\frac{x}{y}$$
   which is a separable equation
   $$y\,dy = -x\,dx$$
   Its solutions are
   $$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + \frac{c}{2}.$$
   In other words, the characteristic curves are
   $$x^2+y^2 = c$$
   which are circles about the origin. The function u is constant on concentric circles, which says the graph $z = u(x,y)$ is rotationally symmetric about the z-axis.

2. *Find the general solution of $x^2\,u_x - xy\,u_y + yu = 0$.*

   The characteristics satisfy
   $$\dot{x} = x^2; \qquad \dot{y} = -xy.$$
   Separate variables in the first equation.
   $$\frac{dx}{x^2} = dt.$$
   The solution of the first with $x(0) = x_0$ is
   $$\frac{1}{x_0} - \frac{1}{x} = \tau$$
   or
   $$x(\tau) = \frac{x_0}{1 - x_0\tau}.$$

Then separating variables in the second equation yields

$$\frac{dy}{y} = -\frac{x_0\, d\tau}{1 - x_0\tau}$$

Hence

$$\log y - \log y_0 = \log(1 - x_0\tau).$$

The solution of the second with $y(0) = y_0$ is

$$y(\tau) = y_0(1 - x_0\tau).$$

so the characteristics are curves that satisfy $xy = \text{const}$.

Let us specify u when the characteristic crosses the line $x = 1$ when $y = x_0y_0$. Starting from there at time zero, at time $t_0 = 1 - 1/x_0$ we flow to $(x(t_0), y(t_0)) = (x_0, y_0)$. Along the characteristic starting at $(1, x_0y_0)$, the solution satisfies

$$\frac{dz}{d\tau} = -yz = x_0y_0(\tau - 1)u$$

so

$$\frac{du}{u} = x_0y_0(\tau - 1)\, d\tau.$$

The solution of this with $u(1, x_0y_0) = u_0$ is

$$\log u - \log u_0 = x_0y_0\left(\frac{1}{2}\tau^2\right)$$

or

$$u\big(x(\tau), y(\tau)\big) = \exp\left(\frac{1}{2}\tau^2 - \tau\right)$$

returning to $t_0 = 1 - 1/x_0$, the general solution is thus

$$\begin{aligned} u(x_0, y_0) = u(x(t_0), y(t_0)) &= f(x_0y_0)\exp\left(\frac{1}{2}\tau_0^2 - \tau_0\right) \\ &= f(x_0y_0)\exp\left(\frac{y_0}{2x_0} - \frac{x_0y_0}{2}\right) = \phi(x_0y_0)\exp\left(\frac{y_0}{2x_0}\right) \end{aligned}$$

where $\phi(w) = f(w)e^{-w/2}$ is an arbitrary function. This general form also works if $x_0 < 0$.

3. *For the transport equation*

$$u_x + 4x^3yu_y = 0$$

find the characteristic curves. Find the general solution. Find the solution associated to the initial condition $u(0, y) = y^2$.

The characteristic curves satisfy

$$\dot{x} = 1, \qquad \dot{y} = 4x^3y.$$

Thus the first has solutions $x(\tau) = x_0 + \tau$. Then the second may be separated

$$d\ln y = \frac{dy}{y} = 4x^3\, d\tau = d\left((x_0 + \tau)^4\right)$$

to yield the solution

$$\ln y - \ln y_0 = (x_0 + \tau)^4 - x_0^4$$

or

$$y(\tau) = y_0 \exp\left((x_0+\tau)^4 - x_0^4\right)$$

The characteristic through the point (x_0, y_0) is the curve

$$y(x) = y_0 \exp\left(x^4 - x_0^4\right).$$

Each of these curves cross the y-axis. Starting from (x_0, y_0) the curve crosses $x = 0$ when $\tau = -x_0$ at the point $y(0) = y_0 \exp\left(-x_0^4\right)$. If we parameterize by the crossing point $y(0) = c$, then the set of points that cross there (the characteristic curve) is given by

$$y = c \exp\left(x^4\right).$$

So the general solution may be assigned any value on the c-characteristic so it is

$$u(x_0, y_0) = g\left(y_0 \exp\left(-x_0^4\right)\right)$$

where $g(z)$ is any function. Checking we see that

$$\begin{aligned} u_{x_0} &= g'\left(y_0 \exp\left(-x_0^4\right)\right) y_0 \exp\left(-x_0^4\right)\left(-4x_0^3\right), \\ u_{y_0} &= g'\left(y_0 \exp\left(-x_0^4\right)\right) \exp\left(-x_0^4\right) \end{aligned}$$

Hence the PDE is satisfied

$$u_{x_0} = -4x_0^3 y_0 u_y.$$

If $u(0, y) = y^2$ then

$$u(x_0, y_0) = y_0^2 \exp\left(-2x_0^4\right).$$

4. *Determine the type of the following PDE. Make a change of variables to bring the equation into canonical form.*

$$2u_{xx} + 4u_{xy} + 5u_{yy} = 0$$

The discriminant is $\Delta = B^2 - 4AC = 16 - 4 \cdot 2 \cdot 5 = -24 < 0$ so the equation is elliptic. We diagonalize the coefficient matrix by changing to eigenvector coordinates. The characteristic polynomial has factorization

$$0 = \begin{vmatrix} 2-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = (2-\lambda)(5-\lambda) - 4 = \lambda^2 - 7\lambda + 6 = (\lambda - 1)(\lambda - 6)$$

An eigenvector for $\lambda = 6$ is found by instepction

$$0 = \begin{pmatrix} 2-\lambda & 2 \\ 2 & 5-\lambda \end{pmatrix} \mathbf{v}_1 = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

For $\lambda = 6$,

$$0 = \begin{pmatrix} 2-\lambda & 2 \\ 2 & 5-\lambda \end{pmatrix} \mathbf{v}_2 = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Making the change to eigen-directions

$$\begin{aligned} \xi &= 2x + y \\ \eta &= -x + 2y \end{aligned}$$

we find

$$\begin{aligned} u_x &= u_\xi\,\xi_x + u_\eta\,\eta_x = 2u_\xi + u_\eta \qquad \text{etc.} \\ u_y &= -u_\xi + 2u_\eta \\ u_{xx} &= 4u_{\xi\xi} + 4u_{\xi\eta} + u_{\eta\eta} \\ u_{xy} &= -2u_{\xi\xi} + 3u_{\xi\eta} + 2u_{\eta\eta} \\ u_{yy} &= u_{\xi\xi} - 4u_{\xi\eta} + 4u_{\eta\eta} \end{aligned}$$

Thus, inserting into the original equation,

$$0 = 2u_{xx} + 4u_{xy} + 5u_{yy} = 5u_{\xi\xi} + 30u_{\eta\eta}$$

A further change to $s = \xi/\sqrt{5}$ and $t = \eta/\sqrt{30}$ reduces the equation to Laplace's equation (the canonical form)

$$0 = u_{ss} + u_{tt}.$$

5. *A flexible chain of length ℓ is hanging from one end $x = 0$ but oscillates horizontally. Let the x-axis point downward and the u-axis point to the right. Assume that the force of gravity at each point of the chain equals the weight of the part of the chain below the point and is directed tangentially along the chain. Assume that the oscillations are small. Find the PDE satisfied by the chain.*

Smallness of the oscillations implies that the motion is entirely horizontal and has zero vertical component. Also that u_x is small and the first order approximation is adequate

$$\sqrt{1 + u_x^2} \approx 1.$$

Compared to the derivation of the usual wave equation, the tensions along the chain are no longer constant. The tension (downward pull force) in the chain is by assumption proportional to the length dangling at point x

$$T(x) = g\rho(\ell - x).$$

Let ρ be the constant mass density per unit length and g the acceleration of gravity. Under the assumptions, the vertical component of the force vectors simply say that the tension on top to hold up a length of string from x_0 to x_1 equals the tension on bottom plus the weight of the length of string, resulting in Newton's Law $ma = F$ to zero acceleration.

$$\begin{aligned} ma &= \frac{T(x_1)}{\sqrt{1 - u_x^2(t, x_1)}} + g\rho(x_1 - x_0) - \frac{T(x_0)}{\sqrt{1 - u_x^2(t, x_0)}} \\ &\approx g\rho(\ell - x_1) + g\rho(x_1 - x_0) - g\rho(\ell - x_0) = 0. \end{aligned}$$

The horizontal component of forces acting on the length of chain is

$$\left. \frac{T(x)u_x}{\sqrt{1 - u_x^2}} \right|_{x_0}^{x_1} = \int_{x_0}^{x_1} \rho u_{tt}\, dx$$

Using smallness, this is approximately

$$g\rho(\ell - x_1)u_x(t, x_1) - g\rho(\ell - x_0)u_x(t, x_0) = \int_{x_0}^{x_1} \rho u_{tt}\, dx$$

Differentiating with respect to x_1 yields the desired PDE

$$g\frac{\partial}{\partial x}\left((\ell - x)\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial t^2}.$$

6. *Solve the initial value problem.*

$$\begin{array}{rcl} \text{(PDE)} & u_t = u_{xx} & \text{for } -\infty \le x \le \infty,\ 0 < t; \\ \text{(IC)} & u(x,0) = \varphi(x) & \text{for } -\infty \le x \le \infty. \end{array}$$

where

$$\varphi(x) = \begin{cases} 1, & \text{if } |x| \le a; \\ 0, & \text{otherwise.} \end{cases}$$

The solution of the initial value problem

$$\begin{array}{rcl} \text{(PDE)} & u_t = u_{xx} & \text{for } -\infty \le x \le \infty,\ 0 < t; \\ \text{(IC)} & u(x,0) = H(x) & \text{for } -\infty \le x \le \infty. \end{array}$$

with the Heaviside Function

$$H(x) = \begin{cases} 0, & \text{if } x < 0; \\ 1, & \text{if } x \ge 0; \end{cases}$$

is given by the special solution found by the similarity method in the text:

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4t}}} e^{-p^2}\,dp = \Phi\left(\frac{x}{\sqrt{4t}}\right)$$

where $\Phi(z)$ is the cumulative distribution function for the standard normal variable. We shall obtain a solution by superposition of reflections and translations of $Q(x,t)$ which preserve the heat equation and make the initial condition.

Observe that $\varphi(x) = H(x+a) - H(x-a)$. The first term turns on when $x > -1$ and the second turns on cancelling the first when $x > a$. Thus the solution of the problem is given by

$$\begin{aligned} u(x,t) = Q(x+a,t) - Q(x-a,t) &= \Phi\left(\frac{x+a}{\sqrt{4t}}\right) - \Phi\left(\frac{x-a}{\sqrt{4t}}\right) \\ &= \frac{1}{\sqrt{\pi}} \int_0^{\frac{x+a}{\sqrt{4t}}} e^{-p^2}\,dp - \frac{1}{\sqrt{\pi}} \int_0^{\frac{x-a}{\sqrt{4t}}} e^{-p^2}\,dp \\ &= \frac{1}{\sqrt{\pi}} \int_{\frac{x-a}{\sqrt{4t}}}^{\frac{x+a}{\sqrt{4t}}} e^{-p^2}\,dp \end{aligned}$$

This is equivalent to the heat kernel formula

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \varphi(y)\,dy = \frac{1}{\sqrt{4\pi t}} \int_{-a}^{a} e^{-\frac{(x-y)^2}{4t}}\,dy$$

7. *Solve*

$$\begin{array}{rll} \text{(PDE)} & \mathcal{L}u = u_{xx} - 3u_{xt} - 4u_{tt} = 0 & \text{for } -\infty \le x \le \infty,\ 0 < t; \\ \text{(IC)} & u(x,0) = x^2, \qquad u_t(x,0) = e^x & \text{for } -\infty \le x \le \infty. \end{array}$$

Factor the operator

$$\mathcal{L}u = (\partial_x + \partial_t)(\partial_x - 4\partial_t)u = 0$$

The solution of the transport equation $(\partial_x + \partial_t)u = 0$ is

$$u(x,t) = f(x-t)$$

and the solution of the transport equation $(\partial_x - 4\partial_t)u = 0$ is

$$u(x,t) = g\left(x + \frac{t}{4}\right).$$

where f, g are any functions. Both solve the equation so by superposition, the most general solution is

$$u(x,t) = f(x-t) + g\left(x + \frac{t}{4}\right)$$

At the initial time,

$$x^2 = u(x,0) = f(x) + g(x)$$

so

$$2x = f'(x) + g'(x).$$

Also

$$u_t(x,t) = -f'(x-t) + \frac{1}{4}g'\left(x + \frac{t}{4}\right)$$

so at the initial time,

$$e^x = u_t(x,0) = -f'(x) + \frac{1}{4}g'(x)$$

The solution of the 2×2 system is

$$f'(x) = \frac{2x - 4e^x}{5}, \qquad g'(x) = \frac{8x + 4e^x}{5}$$

Hence, adding constants of integration,

$$f(x) = \frac{x^2 - 4e^x}{5} + c_1, \qquad g(x) = \frac{4x^2 + 4e^x}{5} + c_2$$

so

$$u(x,t) = \frac{1}{5}(x-t)^2 - \frac{4}{5}e^{x-t} + \frac{1}{20}(4x+t)^2 + \frac{4}{5}e^{x+\frac{t}{4}} + (c_1 + c_2)$$

At time zero,

$$x^2 = u(x,0) = x^2 + (c_1 + c_2)$$

so $c_1 + c_2 = 0$. Thus the solution of the Cauchy Problem is

$$u(x,t) = x^2 + \frac{1}{4}t^2 - \frac{4}{5}e^x\left(e^{-t} - e^{\frac{t}{4}}\right).$$

8. *Let k, a, b, ℓ, M, N and T be positive constants. Suppose $f(x,t)$ is continuous and satisfies $|f(x,t)| \le N$ for all x and t. Let the parabolic boundary $\sqcup_T = \{(0,t) : 0 \le t \le T\} \cup \{(\ell,t) : 0 \le t \le T\} \cup \{(x,0) : 0 \le x \le \ell\}$. Let $u(x,t)$ be continuous on $Q_T = [0,\ell] \times [0,T]$ and be a solution of the heat equation*

$$\text{(HE)} \qquad -u_t + ku_{xx} + au_x - bu = f(x,t) \qquad \text{for } 0 < x < \ell,\ 0 < t \le T;$$

Suppose that u is bounded on the parabolic boundary: $|u(x,t)| \le M$ for all $(x,t) \in \sqcup_T$. Show that $|u(x,t)| \le M + Nt$ for all $(x,t) \in [0,\ell] \times [0,T]$.

We argue that the maximum principle continues to hold for (HE). Let $w(x,t)$ be continuous on $[0,\ell] \times [0,T]$ and satisfy the heat inequality

$$\begin{array}{lrl} \text{(HI)} & -w_t + ku_{xx} + aw_x - bw \ge 0 & \text{for } 0 < x < \ell,\ 0 < t \le T; \\ \text{(IBC)} & w \le 0 & \text{for } (x,t) \in \sqcup_T; \end{array}$$

Then $w \le 0$ on Q_T. To see it, for contradiction there is a point $(x_0, t_0) \in (0,\ell) \times (0,T]$ where w is a positive maximum. There $w(x_0,t_0) > 0$, $w_x(x_0,t_0) = 0$, $w_{xx}(x_0,t_0) \le 0$ and $w_t(x_0,t_0) \ge 0$. But this is a contradiction to (HI):

$$-w_t(x_0,t_0) + ku_{xx}(x_0,t_0) + aw_x(x_0,t_0) - bw(x_0,t_0) \le 0 + 0 + 0 - bw(x_0,t_0) < 0.$$

We prove an upper and a lower inequality for u. First, we consider

$$v(x,t) = u(x,t) - M - Nt$$

Then for $(x,t) \in \sqcup_T$, $v(x,t) \le M - M - Nt \le 0$ which is (IBC), and for $(x,t) \in Q_T$,

$$-v_t + kv_{xx} + av_x - bv = -u_t + N + ku_{xx} + au_x - bu + bM + bNt = f(x,t) + N + bN + bNt \ge 0$$

which is (HI) so $v(x,t) \le 0$ for all $(x,t) \in Q_T$. Similarly, for

$$z(x,t) = -u(x,t) - M - Nt$$

Then for $(x,t) \in \sqcup_T$, $z(x,t) \le M - M - NT \le 0$, and for $(x,t) \in Q_T$,

$$-z_t + kz_{xx} + az_x - bz = u_t + N - ku_{xx} - au_x + bu - bM - bNt = -f(x,t) + N + bN + bNt \ge 0$$

so $z(x,t) \le 0$ for all $(x,t) \in Q_T$. Putting these together imply for $(x,t) \in Q_T$,

$$|u(x,t))| \le M + Nt.$$

9. *Show that solutions of the heat equation exhibit infinite propagation speed.*

Consider the initial value problem for an infinite rod

$$\begin{array}{rcl} \text{(PDE)} & u_t = ku_{xx} & \text{for } -\infty \le x \le \infty,\ 0 < t; \\ \text{(IC)} & u(x,0) = H(x) & \text{for } -\infty \le x \le \infty. \end{array}$$

where the Heaviside Function

$$H(x) = \begin{cases} 0, & \text{if } x < 0; \\ 1, & \text{if } x \ge 0; \end{cases}$$

The solution found by the similarity method is given by

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4t}}} e^{-p^2}\, dp = \Phi\left(\frac{x}{\sqrt{4t}}\right)$$

Note that

$$\lim_{z\to-\infty}\frac{1}{\sqrt{\pi}}\int_0^z e^{-p^2}\,dp = -\frac{1}{2}$$

so that $\Phi(z)\to 0$ as $z\to-\infty$ but $\Phi(z)>0$ for all $z\in\mathbf{R}$ as it is the integral of a positive function.

Now the initial data is positive only for $x\ge 0$ so that the speed of propogation to the point (x,t) with $x<0$ and $t>0$ exceeds $-t/x$ if $Q(x,t)>0$. But $Q(x,t)>0$ for all x and $t>0$ so the propagation speed $-x/t$ may be made as large as you like by decreasing t or sending $x\to-\infty$.

10. *Prove the uniqueness of the wave equation with Neumann boundary conditions.*

$$\begin{array}{rlll}
\text{(PDE)} & u_{tt}=c^2u_{xx} & & \text{for } 0\le x\le \ell,\ 0<t;\\
\text{(IC)} & u(x,0)=\phi(x), & u_t(x,0)=\psi(x) & \text{for } 0\le x\le \ell;\\
\text{(BC)} & u_x(0,t)=a(t), & u_x(\ell,t)=b(x) & \text{for } 0\le t.
\end{array}$$

Suppose there are two solutions $u(x,t)$ and $v(x,t)$. The difference

$$w(x,t)=u(x,t)-v(x,t)$$

satisfies the IBVP

$$\begin{array}{rlll}
\text{(PDE)} & \rho w_{tt}=Tw_{xx} & & \text{for } 0\le x\le \ell,\ 0<t;\\
\text{(IC)} & w(x,0)=0=\tilde\phi(x), & w_t(x,0)=0=\tilde\psi(x) & \text{for } 0\le x\le \ell;\\
\text{(BC)} & u_x(0,t)=0, & u_x(\ell,t)=0 & \text{for } 0\le t.
\end{array}$$

The energy

$$\mathcal{E}(t)=\frac{1}{2}\int_0^\ell \rho w_t^2(x,t)+Tw_x^2(x,t)\,dx$$

starts at

$$\mathcal{E}(0)=\frac{1}{2}\int_0^\ell \rho\tilde\psi^2(x)+T\big(\tilde\phi'(x)\big)^2\,dx=0.$$

It grows according to

$$\begin{aligned}
\frac{d\mathcal{E}}{dt}&=\int_0^\ell \rho w_t\,w_{tt}+Tw_x\,w_{xt}\,dx\\
&=\int_0^\ell \rho w_t\,w_{tt}-Tw_{xx}\,w_t\,dx+\Big[Tw_x\,w_t\Big]_0^\ell\\
&=\int_0^\ell w_t\left(\rho\,w_{tt}-Tw_{xx}\right)\,dx+0-0=0.
\end{aligned}$$

where we have integrated by parts and used (PDE) and (BC). It follows that $\mathcal{E}(t)=\mathcal{E}(0)=0$ is constant. But the only way a continuous function can have

$$0=\frac{1}{2}\int_0^\ell \rho w_t^2(x,t)+Tw_x^2(x,t)\,dx$$

is if $u_t(x,t)=0$ for all t, which implies $x(x,t)=x(x,0)=\tilde\phi(x)=0$ is constant, dead zero. But then $u=v$ so any two solutions have to agree: the problem has a unique solution.