Math 5440 § 1. Treibergs | **First Midterm Exam** | Name: Solutions
Sept. 28, 2016

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1. *Consider the initial value problem where $c > 0$ is constant. Solve the IVP. Make a sketch indicating where the solution is singular. (The solution is singular where it fails to have second derivatives.)*

$$\text{(PDE)} \qquad u_{tt} - c^2 u_{xx} = 0, \qquad \text{for } -\infty < x < \infty \text{ and } 0 < t;$$
$$\text{(IC)} \qquad u(x,0) = \varphi(x) = 0, \qquad \text{for } -\infty < x < \infty;$$
$$u_t(x,0) = \psi(x) = \begin{cases} 1, & \text{if } -\infty < x \le 0; \\ 0, & \text{if } 0 < x. \end{cases}$$

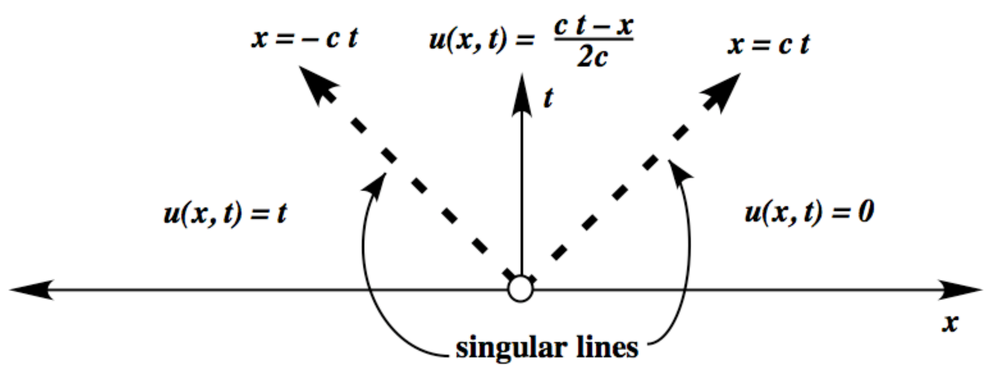
The solution is given by d'Alembert's formula

$$u(x,t) = \frac{1}{2}\Big(\varphi(x-ct) + \varphi(x+ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

Substituting φ and ψ,

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds = \begin{cases} \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} ds = \dfrac{x+ct-(x-ct)}{2c} = t, & \text{if } x+ct \le 0; \\[2ex] \dfrac{1}{2c}\displaystyle\int_{x-ct}^{0} ds = \dfrac{0-(x-ct)}{2c} = \dfrac{ct-x}{2c}, & \text{if } |x| < ct; \\[2ex] \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} 0\,ds = 0, & \text{if } 0 \le x-ct. \end{cases}$$

The solution is singular on the characteristic curves through the singularity of ψ at $x = 0$, namely on $x = \pm ct$.



2. *Suppose that the temperature in a wall of thickness ℓ depends only on the distance x from the outside. Suppose that the wall is insulated on both sides and has an initial temperature profile $f(x)$. Find the steady state temperature that it reaches after a long time. (No heat is gained or lost.) Write the PDE and initial/boundary conditions. Argue that the temperature reaches steady state using the PDE.*

   Let $u(x,t)$ denote the temperature. The temperature satisfies the heat equation with Neumann boundary conditions because the wall is insulated on both sides, where $k > 0$ is the diffusivity constant.

$$
\begin{array}{lll}
\text{(HE)} & u_t = k u_{xx}, & \text{for } 0 < x < \ell \text{ and } 0 < t; \\
\text{(IC)} & u(x,0) = f(x), & \text{for } 0 < x < \ell; \\
\text{(BC)} & u_x(0,t) = u_x(\ell,t) = 0, & \text{for } 0 < t;
\end{array}
$$

   The steady state temperature satisfies no time change in temperature, or $u_t(x,t) = 0$, which implies $u_{xx}(x,t) = 0$ so $u(x,t) = c_1 x + c_2$. With the boundary conditions, $0 = u_x(0,t) = u_x(\ell,t) = c_1$, the steady temperature is constant, $u(x,t) = c_2$.

   The total heat in the rod is invariant.

$$\frac{d}{dt}\int_0^\ell u(s,t)\,ds = \int_0^\ell u_t(s,t)\,ds = k\int_0^\ell u_{xx}(s,t)\,ds = \Big[u_x(x,t)\Big]_{x=0}^{\ell} = 0.$$

   Thus, if the temperature tends uniformly to its limiting temperature

$$\int_0^\ell f(s)\,ds = \int_0^\ell u(s,0)\,ds = \lim_{t\to\infty}\int_0^\ell u(s,t)\,ds = \int_0^\ell c_2\,ds = \ell c_2.$$

   In other words, the limiting temperature is the mean of the initial temperature

$$c_2 = \bar{f} = \frac{1}{\ell}\int_0^\ell f(s)\,ds.$$

   Finally, we can see that the temperature tends to $\bar{f}$ at least in the $\mathcal{L}^2$ sense. Consider the evolution of the $\mathcal{L}^2$-discrepancy. It evolves by

$$
\begin{aligned}
\frac{d}{dt}\int_0^\ell \big(u(s,t)-\bar{f}\big)^2\,ds &= 2\int_0^\ell \big(u(s,t)-\bar{f}\big)u_t(s,t)\,ds \\
&= 2k\int_0^\ell \big(u(s,t)-\bar{f}\big)u_{xx}(s,t)\,ds \\
&= 2k\Big[\big(u(s,t)-\bar{f}\big)u_x(x,t)\Big]_{x=0}^{\ell} - 2k\int_0^\ell u_x^2(s,t)\,ds \\
&= -2k\int_0^\ell u_x^2(s,t)\,ds
\end{aligned}
$$

   Thus, as long as $u(x,t)$ is not constant, $u_x(x,t) \neq 0$ and the $\mathcal{L}^2$-discrepancy decreases. In other words, $u(x,t)$ tends to constant steady state $\bar{f}$ as $t \to \infty$.

   (It's not part of the solution, but it turns out that by using the Wirtinger inequality,

$$\frac{4\pi^2}{\ell^2}\int_0^\ell \big(u(s,t)-\bar{f}\big)^2\,ds \le \int_0^\ell u_x^2(s,t)\,ds,$$

the $\mathcal{L}^2$-discrepancy satisfies

$$\frac{d}{dt}\int_0^\ell \big(u(s,t)-\bar{f}\big)^2\,ds = -2k\int_0^\ell u_x^2(s,t)\,ds \le -\frac{8k\pi^2}{\ell^2}\int_0^\ell \big(u(s,t)-\bar{f}\big)^2\,ds.$$

Integrating the differential inequality yields

$$\int_0^\ell \big(u(s,t)-\bar{f}\big)^2\,ds \le e^{-\frac{8k\pi^2 t}{\ell^2}}\int_0^\ell \big(f(s)-\bar{f}\big)^2\,ds.$$

Thus the $\mathcal{L}^2$-discrepancy tends exponentially to zero as $t\to\infty$.)

3. *What is the type of this PDE? Make a change of variable to reduce the PDE to canonical form. Find the general solution of the PDE.*

$$Lu = u_{xx} + 3u_{xy} + 2u_{yy} = 0$$

The discriminant is

$$\Delta = b^2 - 4ac = 3^2 - 4\cdot 1\cdot 2 = 1 > 0$$

so this PDE is of hyperbolic type. Factoring the operator, we find

$$Lu = (\partial_x + 2\partial_y)(\partial_x + \partial_y)u = 0.$$

We wish to change variables so that

$$\begin{aligned}\partial_\xi &= \partial_x + 2\partial_y\\ \partial_\eta &= \partial_x + \partial_y.\end{aligned}$$

Then, from the chain rule,

$$\begin{aligned}u_\xi &= u_x x_\xi + u_y y_\xi\\ u_\eta &= u_x x_\eta + u_y y_\eta\end{aligned} \implies \begin{aligned}x_\xi &= 1, \quad y_\xi = 2,\\ x_\eta &= 1, \quad y_\eta = 1,\end{aligned}$$

which gives the change of variables

$$\begin{aligned}x &= \xi + \eta\\ y &= 2\xi + \eta\end{aligned} \qquad \text{so inverting,} \qquad \begin{aligned}\xi &= -x + y\\ \eta &= 2x - y.\end{aligned}$$

In these new variables, we get the canonical form $u_{\xi\eta} = 0$. Checking, the derivatives give

$$\begin{aligned}u_x &= u_\xi \xi_x + u_\eta \eta_x = -u_\xi + 2u_\eta,\\ u_y &= u_\xi \xi_y + u_\eta \eta_y = u_\xi - u_\eta,\\ u_{xx} &= u_{\xi\xi} - 4u_{\xi\eta} + 4u_{\eta\eta},\\ u_{xy} &= -u_{\xi\xi} + 3u_{\xi\eta} - 2u_{\eta\eta},\\ u_{yy} &= u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}.\end{aligned}$$

Finally,

$$\begin{aligned}Lu &= u_{xx} + 3u_{xy} + 2u_{yy}\\ &= (u_{\xi\xi} - 4u_{\xi\eta} + 4u_{\eta\eta}) + 3(-u_{\xi\xi} + 3u_{\xi\eta} - 2u_{\eta\eta}) + 2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})\\ &= 0\cdot u_{\xi\xi} + u_{\xi\eta} + 0\cdot u_{\eta\eta} = u_{\xi\eta}\end{aligned}$$

The general solution of $Lu = 0$ is

$$u(\xi,\eta) = f(\xi) + g(\eta)$$

where f and g are arbitrary functions. Returning to the original variables, the general solution is

$$u(x,y) = f(-x+y) + g(2x-y).$$

Checking,

$$u_{xx} + 3u_{xy} + 2u_{yy} = \big(f''(-x+y) + 4g''(2x-y)\big) + 3\big(-f''(-x+y) - 2g''(2x-y)\big) \\ + 2\big(f''(-x+y) + g''(2x-y)\big) = 0.$$

4. *A model for the spread of a disease is given by Fisher's Equation. Its solution $u(x,t)$ gives the proportion who have the disease at location x and time t. Let $\varphi(x)$ be the proportion of the population who have the disease at point x at the initial time, so $0 \le \varphi(x) \le 1$. Show that the solution satisfies $u(x,t) \le 1$ for $(x,t) \in (0,\ell) \times (0,T]$.*

$$\begin{array}{rll} \text{(PDE)} & u_t = u_{xx} + (1-u)u, & \text{for } (x,t) \in (0,\ell) \times (0,T]; \\ \text{(IC)} & u(x,0) = \varphi(x), & \text{for } x \in [0,\ell]; \\ \text{(BC)} & u(0,t) = u(\ell,t) = 0, & \text{for } t \in [0,T]. \end{array}$$

Fisher's Equation is nonlinear, parabolic, so is not exactly one that we've proved the maximum principle for. However, the maximum principle proof technique will give the desired result.

To show $u \le 1$, we assume the contradiction, that there is a point $(x_0,t_0) \in [0,\ell] \times [0,T]$ where the maximum occurs and exceeds the bound $u(x_0,t_0) > 1$. But by (IC) and (BC), $u(x,t) \le 1$ on $t = 0$ or $x = 0$ or $x = \ell$ so we must have $(x_0,t_0) \in (0,\ell) \times (0,T]$ in the interior or $t = T$. At this maximum point, $u_x = 0$, $u_{xx} \le 0$ and $u_t \ge 0$. However, this leads to a contradiction to the (PDE),

$$0 \le u_t(x_0,t_0) = u_{xx}(x_0,t_0) + \big(1 - u(x_0,t_0)\big)u(x_0,t_0) \le 0 + \big(1 - u(x_0,t_0)\big)u(x_0,t_0) < 0$$

because $u(x_0,t_0) > 1$. Hence we conclude $u(x,t) \le u(x_0,t_0) \le 1$ for all $(x,t) \in [0,\ell] \times [0,T]$.

5. *Consider the transport equation with decay. Find the characteristic curves. Change variables and reduce (TED) to an ODE with initial condition. Solve the ODE and write the general solution of (TED). Find the particular solution that satisfies the initial condition with $\varphi(x) = e^{-x}$.*

$$\begin{array}{rll} \text{(TED)} & u_t + 3u_x = -2u, & \text{for } -\infty < x < \infty \text{ and } 0 < t. \\ \text{(IC)} & u(x,0) = \varphi(x), & \text{for } -\infty < x < \infty. \end{array}$$

The characteristic curves satisfy

$$\dot{t} = 1, \qquad \dot{x} = 3, \qquad \text{so} \qquad \frac{dx}{dt} = 3.$$

The solution through the point (x_0,t_0) is $x = 3(t - t_0) + x_0$. Each of these curves pass through a different point on the axis $(x_0, 0)$ so the all the characteristic curves are given by the equations

$$x = 3t + x_0.$$

where x_0 ranges through all real numbers. Change variables to $t = \xi$ and $x = 3\xi + \eta$. Then

$$u_\xi = u_t t_\xi + u_x x_\xi = u_t + 3u_x = -2u$$

with initial condition $u(x,t) = u(\eta,0) = \varphi(\eta)$ when $\xi = 0$. Thus the solution is

$$u(\xi,\eta) = \varphi(\eta)e^{-2\xi}.$$

Returning to the original variables, the general solution is

$$u(x,t) = \varphi(x-3t)e^{-2t}.$$

where φ is an arbitrary function. Checking

$$u_t + 3u_x + 2u = e^{-2t}\big(-2\varphi(x-3t) - 3\varphi'(x-3t) + 3\varphi'(x-3t) + 2\varphi(x-3t)\big) = 0.$$

When the initial condition is $\varphi(\eta) = e^{-\eta}$, the particular solution is

$$u(x,t) = \varphi(x-3t)e^{-2t} = e^{-(x-3t)}e^{-2t} = e^{t-x}.$$