Math 5470 § 1. **Second Midterm Exam** Name: Practice Problems
Treibergs March 22, 2016

1. *Marvel's model of public opinion considers four nonoverlapping groups in a population. Let p be the unchanging fraction of true believers in opinion A. Let x denote the fraction sympathetic but uncommitted. If someone argues an opposing opinion B, then these people become an AB, seeing the merits of both positions. Likewise, let y be the fraction of B sympathetic but uncommitted believers who will become AB upon hearing A argued. The AB population does not try to convert anyone but will join the A or B camp upon hearing such an argument. At every instant, two random people are chosen, an advocate and a listener. Let $n_{AB} = 1 - p - x - y$. Explain the equations. Assuming that initially everyone believes B, $x(0) = 0$, $n_{AB}(0) = 0$ so $y(0) = 1 - p$. Let the dynamics come to equilibrium. Show that the final state changes abruptly: for small p most people accept B but for $p > p_c$ most accept A. Find p_c. What kind of bifurcation occurs there? [Note: this was inspired by Strogatz's problem 8.1.15. However, there is a typo in the equations.]*

$$\dot{x} = (p + x)n_{AB} - xy$$
$$\dot{y} = n_{AB}y - (p + x)y$$

To explain the equations, note that the population is split into four factions $1 = p + x + y + n_{AB}$ and since we assume that listeners and advocates encounter each other randomly, the probability of an encounter is given by the product of proportions. Let us present the matrix of possible encounters and indicate their effect on the x, y populations, resp.

Listeners\Advocates	P	X	N	Y
P	0,0	0,0	0,0	0,0
X	0,0	0,0	0,0	-,0
N	+,0	+,0	0,0	0,+
Y	0,-	0,-	0,0	0,0

For example, if y hears p then there is no change in x but y switches to n_{AB} and decreases y. Also is N_{AB} does not advocate so both its columns are zero. Thus each equation has three terms, with signs as in the table.

Replacing n_{AB} in the equations yields the system

$$\begin{aligned}
\dot{x} &= (p + x)(1 - p - x - y) - xy &&= p(1 - p) - 2px - py + x - x^2 - 2xy &&= f(x, y)\\
\dot{y} &= (1 - p - x - y)y - (p + x)y &&= y - 2py - 2xy - y^2 &&= g(x, y)
\end{aligned}$$

The quantities are nonnegative so $0 \le x$, $0 \le y$ and $0\,\mathrm{length}_{AB}$ which implies $x + y \le 1 - p$, thus we're interested in values in this triangle $\mathcal{T}$ only. $\dot{y} \ge 0$ at $y = 0$, $\dot{x} = p(1 - p - y)$ which is nonnegative for $0 \le y \le 1 - p$ and $\dot{x} + \dot{y} = -(p + 2x)y \le 0$ on $x \ge 0$ and $y \ge 0$, so $\mathcal{T}$ is a trapping region.

Let's find the fixed points. $g = 0$ implies $y = 0$ or $2x + y = 1 - 2p$. If $y = 0$ then $f = 0$ implies $(p + x)(1 - p - x) = 0$ which means $x = 1 - p$ is the only solution in $\mathcal{T}$. If $y \ne 0$ and $y = 1 - 2p - 2x$, then $f = 0$ implies

$$y = \frac{(p + x)(1 - p - x)}{p + 2x}$$

Equating y yields $3x^2 - (1 - 4p)x + p^2 = 0$ whose root in $\mathcal{T}$ are

$$x_\pm^* = \frac{1 - 4p \pm \sqrt{(1 - 4p)^2 - 12p^2}}{6}, \quad y_\pm^* = 1 - 2p - 2x_\pm^* = \frac{2 - 2p \mp \sqrt{1 - 8p + 4p^2}}{3}$$

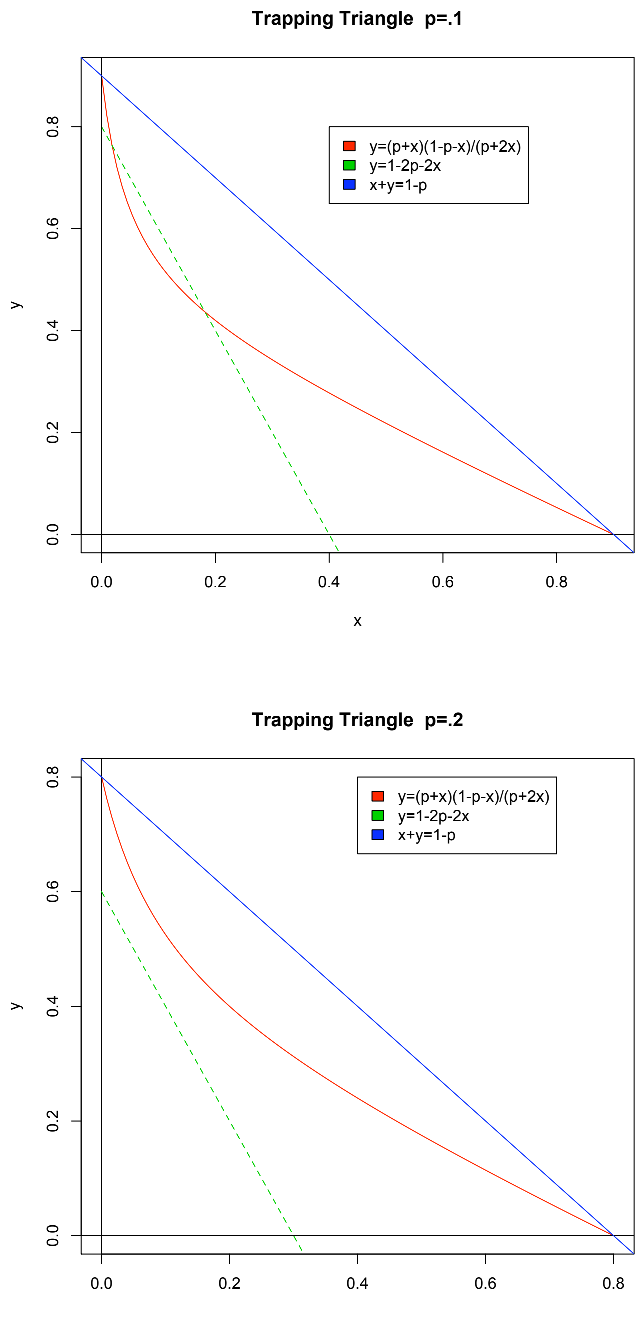


Figure 1: **R**© plots of isoclines for $p = .1$ and $p = .2$.

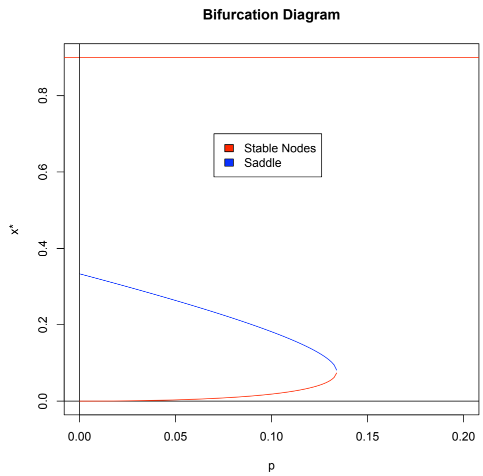


Figure 2: **R**© Bifurcation Diagram.

The radicand $1 - 8p + 4p^2$ is nonnegative for $0 \le p < p_c$ where $p_c = 1 - \frac{\sqrt{3}}{2} = 0.1339746$ and negative if $p_c < p \le 1$.

We compute the stability at the three rest points. The Jacobian matrix is

$$J(x,y) = \begin{pmatrix} 1-2p-2x-2y & -p-2x \\ -2y & 1-2p-2x-2y \end{pmatrix}$$

At $(1-p,0)$ it is

$$J(1-p,0) = \begin{pmatrix} -1 & p-2 \\ 0 & -1 \end{pmatrix}$$

which is a stable degenerate node. At $(x^*_\pm, y^*_\pm)$, using $y^*_\pm = 1 - 2p - 2x^*_\pm$

$$J(x,y) = \begin{pmatrix} 1-2p-2x-2y & -p-2x \\ -2y & 1-2p-2x-2y \end{pmatrix} = \begin{pmatrix} -y^* & y^*+p-1 \\ -2y^* & -y^* \end{pmatrix}$$

so the trace is $\tau = -2y^* < 0$ and the determinant is

$$\Delta = y^*(3y^* + 2p - 2) = \mp y^* \sqrt{1 - 8p + 4p^2}$$

Thus in the range $0 < p < p_c$, the radicand is positive and (x^*_+, y^*_+) is a saddle and (x^*_-, y^*_-) is a stable node. A saddle-node bifurcation happens at $p = p_c$.

As long as $p < p_c$ there is a stable node at (x^*_-, y^*_-) that attracts all initial points on the y-side of the stable manifold W^s of (x^*_+, y^*_+). But as p increases through $p_c \approx 13\%$, the saddle and node collide and annihilate, and there is a catastrophe with $(1-p,0)$ the surviving globally stable node, which attracts all public opinion.

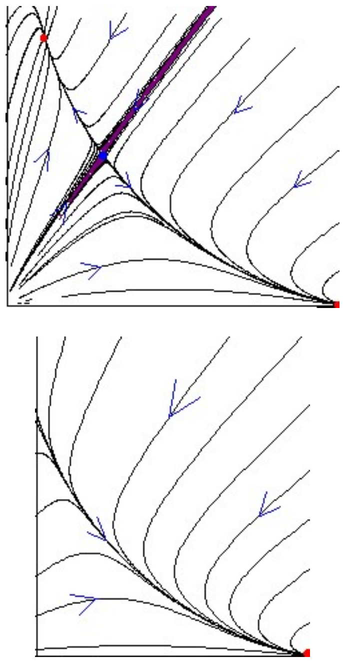


Figure 3: Phase portraits for $p = .1$ and $p = .2$ plotted using 3D-XplorMath©.

2. *Find the fixed points and classify them. Show that $x = y$ is an invariant line. Show that $|x(t) - y(t)| \to 0$ for all other trajectories. Sketch the phase portrait. Observe that solutions approach a certain curve as $t \to -\infty$. Can you explain this intuitively and suggest an approximate equation for this curve? [Strogatz, 6.3.9.]*

$$\dot{x} = y^3 - 4x$$
$$\dot{y} = y^3 - y - 3x$$

To find the fixed points, $f = 0$ says $x = y^3/4$ so $g = 0$ implies $y(y^2 - 4) = 0$ so $y = 0, \pm 2$ and $x = 0, \pm 2$, resp. The Jacobian is

$$J(x,y) = \begin{pmatrix} -4 & 3y^2 \\ -3 & 3y^2 - 1 \end{pmatrix}$$

At the origin

$$J(0,0) = \begin{pmatrix} -4 & 0 \\ -3 & -1 \end{pmatrix}$$

is a stable node. The slow $\lambda_1 = -1$ incoming direction is $(1, 0)$ and the fast $\lambda_1 = -4$ incoming direction is $(1, 1)$. Most trajectories come in parallel to the slow direction in the linearized flow. The nonlinear terms are much smaller at the origin so do not affect the incoming direction. At the origin

$$J(\pm 2, \pm 2) = \begin{pmatrix} -4 & 12 \\ -3 & 11 \end{pmatrix}$$

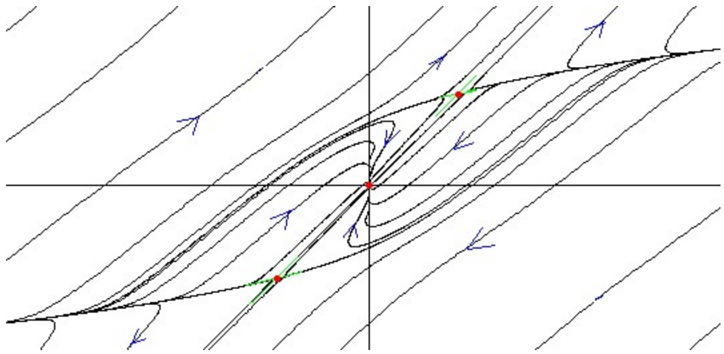


Figure 4: Phase portrait using 3D-XplorMath©.

The determinant is $\Delta = -8$ which is a saddle. The eigenvector for $\lambda = 1$ is $(1, 1)$ and for $\lambda = -8$ is $(4, 1)$.

To see that the line $x = y$ is invariant and that $|x(t) - y(t)| \to 0$ for all trajectories, consider $w = x - y$.

$$\dot{w} = \dot{x} - \dot{y} = (y^3 - 4x) - (y^3 - y - 3x) = -x - y = -w$$

The solution is $w(t) = (x_0 - y_0)e^{-t}$. Thus if $x_0 - y_0 = 0$ then $w(t) = 0$ for all time: the flow stays on the line $x = y$. On the other hand

$$|x(t) - y(t)| = |w(t)| = |x_0 = y_0|e^{-t} \to 0$$

as $t \to \infty$.

What is the curve when $t \to -\infty$? The time-reversed equations are

$$\dot{x} = -y^3 + 4x$$
$$\dot{y} = -y^3 + y + 3x$$

The solution moves quickly as $t \to \infty$ in the reversed equations, until the y motion slows down. That happens approximately when $-y^3 + 4x = 0$ or $y = (4x)^{1/3}$. Then y motion is slow, but

$$\dot{x} = -y^3 + 3x + y = -y^3 + 4y + (x - y) \approx 0 + w$$

grows exponentially. (The better behaved time-reversed equations were used to make the plot.)

3. *Is the origin a nonlinear center? [Strogatz 6.6.10.]*

$$\dot{x} = -y - x^2$$
$$\dot{y} = x$$

The system is reversible in the following more general sense: it is invariant under $t \mapsto -t$ and $x \mapsto -x$. Indeed, if we put $\xi(\tau) = -x(-t)$ then $\xi' = \dot{x} = -y - x^2 = -y - (-\xi)^2$ and $y' = -\dot{y} = -x = \xi$. We may apply the theorem on centers of reversible systems because both f and g are continuously differentiable (they are polynomials!) If the linearization is a center, then so is the nonlinear system near the origin. The Jacobian is

$$J(x, y) = \begin{pmatrix} -2x & -1 \\ 1 & 0 \end{pmatrix}, \qquad J(0, 0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

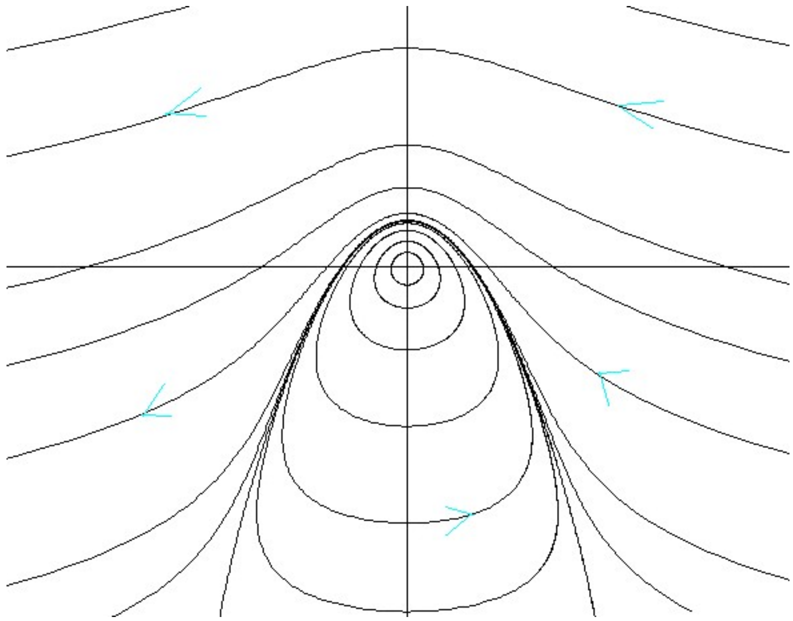


Figure 5: Reversible system with center using 3D-XplorMath©.

whose eigenvalues are $\pm i$. Thus the linearization at the origin has centers, so by the theorem, so does the nonliniear system near the origin.

4. *Can a fixed point be Liapunov stable but not attracting? Can a fised point be attracting but not Liapunov stable? Give examples and explain.*

   Let's give examples in $\mathbf{R}^2$. Consider linear centers $\ddot{x} + x = 0$ or

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -x\end{aligned}$$

The general solution is $x(t) = A\cos t + B\sin t$ and $y(t) = \dot{x}(t) = -A\sin t + B\sin t$, which are circles centered at the origin. The origin is Liapunov stable but not attracting. Indeed, if $(x_0, y_0) \neq (0,0)$ then $(x(t), y(t))$ does not converge to $(0,0)$ as $t \to \infty$, so the origin is not attractive. It is Liapunov stable. We have to verify the definition of Liapunov stability.

Definition 1. *Suppose we have a dynamical system in an open set $\Omega \in \mathbf{R}^2$ given by a differential system*

$$\begin{cases} \dot{x} = f(x,y) & x(0) = x_0 \\ \dot{y} = g(x,y) & y(0) = y_0 \end{cases}$$

where $f, g \in \mathcal{C}^1(\Omega)$ and $(x_0, y_0) \in \Omega$ is any initial point. We say a rest point $(x^*, y^*) \in \Omega$ is Liapunov Stable if for every $\epsilon > 0$ there is a $\delta > 0$ such that $(x(t), y(t)) \in \Omega$ exists and satisfies $\|(x(t), y(t)) - (x^*, y^*)\| < \epsilon$ for all $t \geq 0$ whenever $(x_0, y_0) \in \Omega$ satisfies $\|(x_0, y_0) - (x^*, y^*)\| < \delta$.

To show that the centers are Liapunov stable, choose $\epsilon > 0$. Let $\delta = \epsilon$. Then whenever $\|(x_0, y_0) - (0,0)\| < \delta$ then $\|(x(t), y(t)) - (0,0)\| = \|(x_0, y_0) - (0,0)\| < \delta = \epsilon$ so $(0,0)$ is Liapunov stable.

An example of an attractive system that is not Liapunov stable is given by the critical system in the infinite period bifurcation.

$$\dot{r} = r(1 - r^2)$$
$$\dot{\theta} = 1 - \cos\theta$$

Note that $(r^*, \theta^*) = (1, 0)$ is a rest point. Along the ray $\theta_0 = 0$ there is no angular change but if $r_0 > 0$, $(r(t), \theta(t)) \to (1, 0)$. Away from this ray, $\dot{\theta} > 0$ so if $r_0 > 0$ and $0 < \theta_0 < 2\pi$ then θ increases to 2π taking infinite time. In the meanwhile $r(t) \to 1$ as $t \to \infty$ so $(1, 2\pi)$ is attractive. However, it is not Liapunov stable. Indeed, for any $0 < \epsilon < 1$, flow does not stay in the ϵ-neighborhood of $(1, 0)$ no matter how close (r_0, θ_0) is to $(1, 0)$. For any $0 < \delta < 1$ take $r_0 = 1$ and $0 < \theta_0 < \delta/2$ so $\|(r_0, \theta_0) - (1, 0)\| < \delta$. The point $(r(t), \theta(t))$ flows around the circle and at some $t_1 > 0$ crosses the point is diametrically opposite the initial point $(r(t_1), \theta(t_1)) = (1, \pi)$ which is not in the ϵ neighborhood of $(0, 1)$.

5. *Find the index of the vector field with respect to the unit circle $x^2 + y^2 = 1$.*

$$\dot{x} = x^2 + y^2 \qquad = f(x, y)$$
$$\dot{y} = x^2 - y^2 \qquad = g(x, y)$$

Let $(\cos t, \sin t)$ run through the unit circle C as $0 \le t \le 2\pi$. Note that on the circle we have $f(x, y) = 1$ for all points on C. This means that the vector field $V = (f, g)$ alway has a positive component in the positive x direction, and doesn't wind around the origin. Its angle with $\mathbf{e}_1$ is

$$\phi(t) = \angle(V(t), \mathbf{e}_1) = \operatorname{Atn}\left(\frac{g(\cos t, \sin t)}{f(\cos t, \sin t)}\right) = \operatorname{Atn}(\cos^2 t - \sin^2 t) = \operatorname{Atn}(\cos 2t)$$

which is a periodic function. Thus the index is

$$\operatorname{ind}_C(V) = \frac{1}{2\pi}(\phi(2\pi) - \phi(0)) = 0.$$

6. *Consider the equation*

$$\ddot{x} = x - x^2$$

Find a conserved quantity for the system Find and classify the equilibrium points. Sketch the phase portrait. Find the equation for the homoclinic orbit that separates the closed and non-closed trajectories.

Multiplying by $\dot{x}$

$$\ddot{x}\dot{x} - x\dot{x} + x^2\dot{x} = \left(\frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3\right)' = 0$$

and integrating gives the desired conserved quantity

$$W = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3$$

Putting $y = \dot{x}$ lets us write the system

$$\dot{x} = f(x, y) = y$$
$$\dot{y} = g(x, y) = x - x^2$$

The $\dot{x} = y = 0$ isocline is $y = 0$ (Red line in Fig. 1) where flow is vertical. Above the axis, flow is to the right, below to the left. The $\dot{y} = x - x^2 = 0$ isoclines are the two (Blue) lines

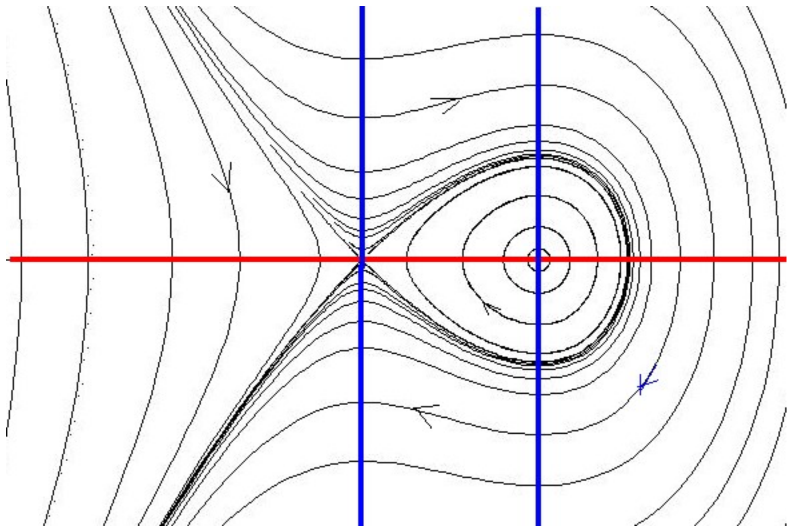


Figure 6: Nullclines and Trajectories from 3D-XplorMath©.

$x = 0$ and $x = 1$ where flow is horizontal. between the lines, flow is up, outside is down. The rest points are intersections of the isoclines, $(0, 0)$ and $(1, 0)$.

The Jacobian is

$$J(x,y) = \begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial x} \\ \dfrac{\partial g}{\partial y} & \dfrac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1-2x & 0 \end{pmatrix}$$

At the rest point $(0, 0)$,

$$J(x,y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

has determinant $\Delta = -1$ and trace $\tau = 0$ which is a saddle. Since the eigenvalues add to zero, they are $\lambda_1 = 1$ and $\lambda_2 = -1$. Since they have nonzero real parts, the behavior of the linear and nonlinear flows near $(0, 0)$ are conjugate: the flow is a saddle. At the rest point $(1, 0)$,

$$J(x,y) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

has determinant $\Delta = -1$ and trace $\tau = 0$ which is a center. The eigenvalues are $\lambda_1 = i$ and $\lambda_2 = -i$. Thus the Hartman-Grobman Theorem does not apply, and we cant be sure that the nonlinear flow will al;so have centers. However, since there is a conserved quantity, the trajectories follow level sets $W(x(t), y(t)) = C$. Near $(1, 0)$, substituting $x = (x - 1) + 1$ the conserved quantity is is

$$\begin{aligned} W(x,y) &= \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3 \\ &= \frac{1}{2}y^2 - \frac{1}{2}[(x-1)+1]^2 + \frac{1}{3}[(x-1)+1]^3 \\ &= -\frac{1}{6} + \frac{1}{2}y^2 + \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 \end{aligned}$$

The cubic term is small for x near 1 so this says the trajectories which are level sets of W near $(1,0)$ where $W = -1/6$ are closed curves too.

The homoclinic orbit (from $(0,0)$ to itself) is the $x \ge 0$ part of the level set

$$\frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3 = W(0,0) = -\frac{1}{6}.$$

7. *Show that the following system does not have nontrivial periodic orbits.*

$$\begin{aligned} \dot{x} &= -y - x^3 \\ \dot{y} &= x - y^3 \end{aligned}$$

The only rest point is the origin. To see it, note that $f = 0$ implies $y = -x^3$ so $g = 0$ implies $x = y^3 = -x^9$ or $x(1 + x^8) = 0$ which has only $x = 0$ as the solution. Hence $y = 0$.

Flow of the linear part is circles and for the nonlinear part is inward. It suggests to consider the Liapunov function $V = x^2 + y^2$. Differentiating,

$$\dot{V} = 2x\dot{x} + 2y\dot{y} = 2x(-y - x^3) + 2y(x - y^3) = -2(x^4 + y^4)$$

which is negative away from the origin so V decreases for all time on trajectories starting away from the origin. Thus there cannot be nontrivial (*i.e.,* not a single point) closed orbits because they contain points other than the origin which may be used as starting points from which $V(x(t), y(t))$ decreases forever instead of oscillating as it would do on a periodic trajectory. Incidentally, finding a global strict Liapunov funntion implies that the origin is asymptotically stable with basin of attraction the whole plane.

Another argument may be given by Dulac's criterion. Observe that

$$\operatorname{div}(f,g) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = -3x^2 - 3y^2$$

which is negative away from the origin. Thus, if C is any nontrivial periodic orbit and D the region enclosed by C, then by the divergence theorem,

$$0 > \iint_D \operatorname{div}(f,g)\, da = \int_{\partial D} (f,g) \bullet \nu \, ds = 0$$

is a contradiction. Here $\partial D = \pm C$ is the boundary, which is C taken with the positive orientation and ν is the outward unit normal vector which is perpendicular to the flow direction.

8. In the given system, a Hopf bifurcation occurs at the origin at some critical value of the parameter μ. What is this critical value?

$$\begin{aligned} \dot{x} &= y + \mu x \\ \dot{y} &= -x + \mu y - x^2 y \end{aligned}$$

A Hopf bifurcation occurs if a complex conjugate pair of roots cross the imaginary axis. The Jacobian is

$$J(x,y) = \begin{pmatrix} \mu & 1 \\ -1-2xy & \mu - x^2 \end{pmatrix}; \qquad J(0,0) = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}$$

The eigenvalues are roots of $(\mu-\lambda)^2+1=0$ or $\lambda = \mu \pm i$. Thus the pair crosses the imaginary axis exactly when $\mu = 0$. Linearization at $(0,0)$, says stability changes from stable to unstable as μ increases through zero.

9. *Consider the following system in polar coordinates. The system is known to undergo a bifurcation as the parameter μ passes through a certain critical value μ_c. Find that value and describe the type of bifurcation.*

$$\dot r = r(1+\mu r + r^2)$$
$$\dot\theta = 1 + \sin\theta\cos\theta$$

Note that $\sin\theta\cos\theta = \frac{1}{2}\sin(2\theta) \in [-\frac{1}{2},\frac{1}{2}]$ so that $\frac{1}{2} \le \dot\theta \le \frac{3}{2}$. Hence the flow winds around the origin in a counterclockwise direction at speeds that vary according to angle. Thus the origin is the only fixed point.

The nonegative roots of $r(1+\mu r+r^2)$ are zero and

$$r = \frac{-\mu \pm \sqrt{\mu^2-4}}{2}$$

Hence there are no roots if $|\mu| < 2$ and the roots are negative if $\mu \ge 2$. Thus the bifurcation point is $\mu = -2$, and there are two positive roots if $\mu < -2$. For $\mu = -2$, the root is $r_\pm = 1$, which is an periodic orbit which is semistable: it is approached by flow from the inside and flow spirals away on the outside. For $\mu < -2$, there are two roots $0 < r_- < r_+$. $\dot r$ is negative between the roots and positive outside. Thus $r_\pm$ correspond to two limit cycles, the inner one is stable as flow approaches it from outside and inside, and the outer one is stable since trajectories leave it inside and outside, in other words, it is stable for backward flow.

Thus the bifurcation is a saddle-node bifurcation of cycles. As the parameter is increased through -2, two circular cycles, a stable inner one and an unstable outer one merge into a semistable cycle at $\mu = -2$ and $r = 1$ and then vanishes leaving only an outward spiral in the whole plane.

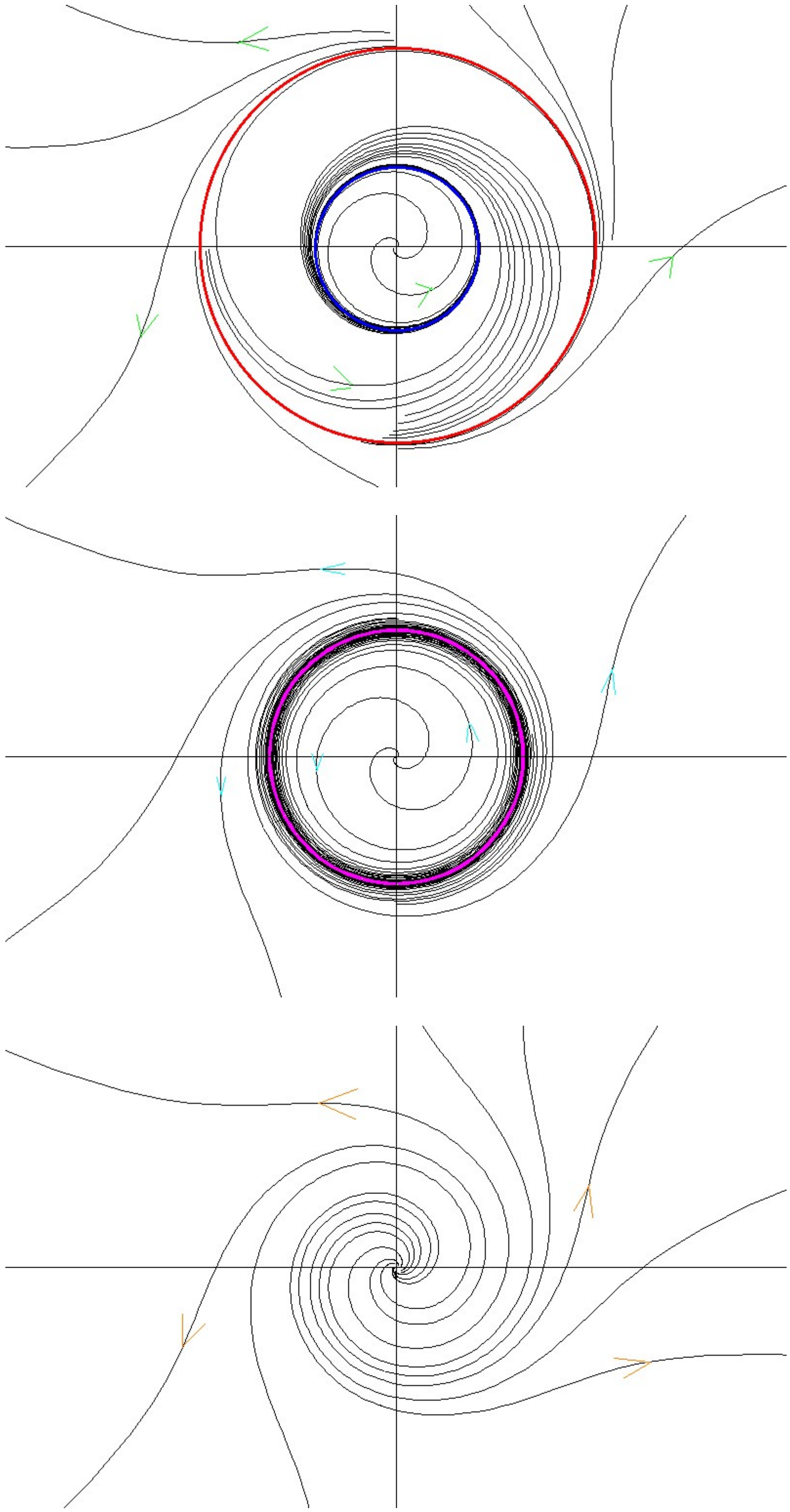
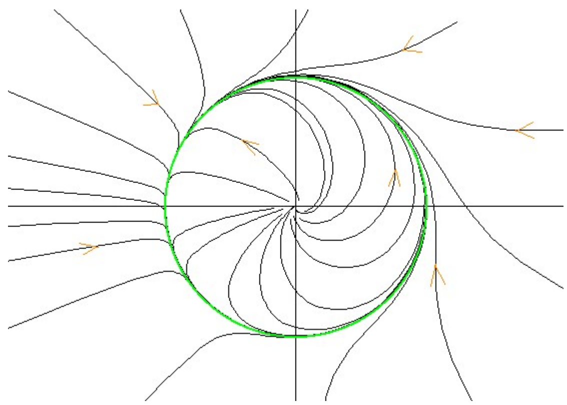


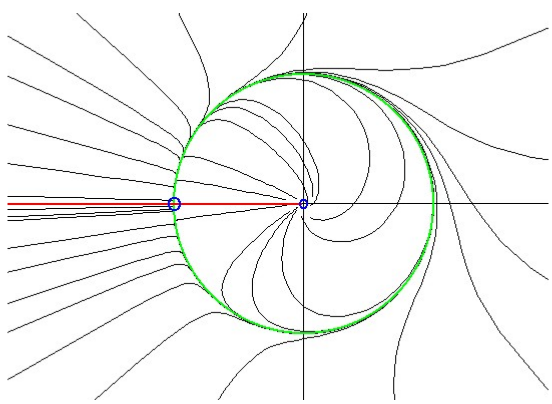
Figure 7: Limit cycles : two ($\mu = . - 2.2$), one ($\mu = -2$), none ($\mu = -1.8$) [3D-XplorMath©]

10. *Consider the following system in polar coordinates. The system is known to undergo a bifurcation as the parameter $\mu \geq 0$ passes through a certain critical value μ_c. Find that value and describe the type of bifurcation.*

$$\dot{r} = r(1 - r^2)$$
$$\dot{\theta} = 1 + \mu \cos\theta$$

The system has an unstable rest point at the origin and trajectory at $r = 1$. For $\mu < 1$ we have $\dot{\theta} > 0$ and $r = 1$ is a stable limit cycle. But at $\mu \geq 1$ there are two solutions of $1 + \mu \sin\theta = 0$, namely $\pi/2 < \theta_- \leq 3\pi \leq \theta_+ < 3\pi/2$. The rays $\theta = \theta_\pm$ are invariant: flow is toward $r = 1$ along these rays. For $\theta_- < \theta < \theta_+$ the angular direction reverses sonce $\dot{\theta} < 0$ there. Thus $(r, \theta) = (1, \theta_-)$ is a stable rest point and $(1m\theta_+)$ is a saddle. This is an infinite period bifurcation: as μ increases throught $\mu_c = 1$, flow on the stable limit cycle slows until it stops at $\theta = \pi$, when the homoclinic orbit from $(1, \pi)$ along $r = 1$ to itself takes infinite time, and then a stable node and saddle form.





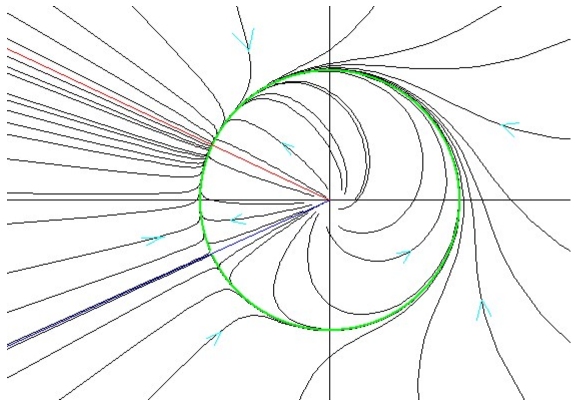


Figure 8: Infinite time bifurcation : $\mu = .8$, $\mu = .1$, $\mu = 1.2$. [3D-XplorMath©]

11. *Find the value of the parameter where the system undergoes a homoclinic bifurcation. Sketch the phase portrait just above and below the bifurcation.*

$$\ddot{x} - (6x^2 - 4x^3 - 6\dot{x}^2 + 12\mu)\dot{x} - x + x^2 = 0$$

The corresponding system is

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= x - x^2 + (6x^2 - 4x^3 - 6y^2 + 12\mu)y\end{aligned}$$

The rest points are at $y = 0$ and $x - x^2 = 0$ so at $x = 0, 1$. If the y term were not in the equation then there is a conserved quantity

$$V = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3$$

Is it conserved for some choices of μ? Checking its evolution in the full equation,

$$\begin{aligned}\dot{V} &= y\dot{y} - (x - x^2)\dot{x} &(1)\\ &= y\big[x - x^2 + (6x^2 - 4x^3 - 6y^2 + 12\mu)y\big] - (x - x^2)y &(2)\\ &= (6x^2 - 4x^3 - 6y^2 + 12\mu)y^2 &(3)\\ &= 12(\mu - V)y^2 &(4)\end{aligned}$$

Thus $\dot{V}$ is nonnegative if $V < \mu$, zero if $V = \mu$ and nonpositive if $V < \mu$. In case $\mu = 0$ there is a homoclinic loop

$$\Gamma_0 = \{(x, y) : V(x, y) = 0,\ x \geq 0\}$$

which passes through $(0, 0)$ and $(1.5, 0)$ where the flow is downward so clockwise in Γ_0. Points outside the loop have $V > 0$ so V is decreasing unless $y = 0$ but even then flow doesn't stop off the rest points so V continues to decrease. Inside the loop $V < 0$ so similarly V is increasing. Thus the homoclinic orbit is attracting. The Jacobian is

$$J(x, y) = \begin{pmatrix} 0 & 1 \\ 1 - 2x + 12xy - 12x^2y & 6x^2 - 4x^3 - 18y^2 + 12\mu \end{pmatrix}$$

so

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ \\ 1 & 12\mu \end{pmatrix}$$

whose eigenvalues are $6\mu \pm \sqrt{36\mu^2+1}$ with eigenvectors $(1, 6\mu \pm \sqrt{36\mu^2+1})$, resp. For $\mu = 0$ this is $\lambda_\pm = \pm 1$ with eigenvectors $(1, \pm 1)$, resp. which is a saddle.

When $-\frac{1}{6} < \mu < 0$ then the level curve

$$\Gamma_\mu = \{(x,y) : V(x,y) = \mu,\ x > 0\}$$

is invariant and is a closed loop inside Γ_0. It does not contain fixed points so is a periodic orbit. If $(x-0, y_0)$ is a point with $x_0 > 0$ and $-\frac{1}{6} < V(x_0,y_0) < \mu$ then $\dot{V} \geq 0$ and $(x(t), y(t))$ tends to Γ_μ as $t \to \infty$. Similarly, if $x_0 > 0$ and $\mu < V(x_0,y_0) < 0$ then $\dot{V} \leq 0$ then $((x(t), y(t))$ is trapped inside $V^{-1}(0)$ and $(x(t), y(t))$ spirals inward to Γ_μ as $t \to \infty$. In particular, the $(1, 6\mu + \sqrt{36\mu^2+1})$ branch of the unstable manifold starting at the origin spirals down to Γ_μ. If (x_0, y_0) with $x_0 > 0$ is on the stable manifold of the origin then $(x(t), y(t)) \to (0,0)$ as $t \to \infty$. But running time backward, $(x(t), y(t))$ goes outside the unstable manifold at the origin and continues to the positive y-axis and then off to infinity as $t \to -\infty$. In particular, there is no longer a homoclinic loop from W^u at the origin back to W^s at the origin. As $\mu \to 0$ the limit cycle Γ_μ grows passing closer and closer to the fixed point at the origin; it takes longer and longer to pass by the fixed point $(0,0)$ and so the period goes to infinity.

Next we consider $\mu > 0$. The invariant curve $V^{-1}(\mu)$ is unbounded without closed loops. In fact, we argue that there are no closed periodic orbits. For (x_0, y_0) on W^u near the origin, $V(x_0, y_0)$ is close to zero and less than μ. $\dot{V} \geq 0$ and increases since the trajectory does not stop at $y = 0$. The trajectory cannot close and so passes outside W^s at the origin, thus the homoclinic loop is destroyed. Thus if (x_0, y_0) is any point within Γ_0 which is not a rest point (*i.e.*, $x_0 > 0$ and $-\frac{1}{6} < V(x_0, y_0) < 0$ then (x_0, y_0) is either on the stable manifold of the origin or eventually the trajectory winds its way out of Γ_0 and shoots off to infinity as t increases. $\dot{V} > 0$ on Γ_0 so flow is outward. Similar considerations show that points starting outside Γ_0 also tend to infinity. Thus any point in the plane, not a rest point or on the stable manifold of the origin must eventually shoot off to infinity as t increases.

[Problem from R. Clark Robinson, *An Introduction to Dynamical Systems Continuous and Discrete,* 2004, Pearson Education Inc., pp. 203–205.]

12. *Consider the system where a and b are parameters* $(0 < a \leq 1,\ 0 \leq b < \frac{1}{2})$. *Rewrite in polar coordinates. Prove that there is at least one limit cycle, and if there are several, they all have the same period $T(a,b)$. Prove that for $b = 0$ there is only one limit cycle. [Strogatz, Problem 7.3.7.]*

$$\begin{aligned} \dot{x} &= y + ax(1 - 2b - x^2 - y^2) \\ \dot{y} &- x + ay(1 - x^2 - y^2) \end{aligned}$$

Let $x = r\cos\theta$ and $y = r\sin\theta$ so $x^2 + y^2 = r^2$. Substituting,

$$\begin{aligned} \dot{r}\cos\theta - r\dot{\theta}\sin\theta &= r\sin\theta + ar(1 - 2b - r^2)\cos\theta \\ \dot{r}\sin\theta + r\dot{\theta}\cos\theta &- r\cos\theta + ar(1 - r^2)\sin\theta \end{aligned}$$

Multiplying one equation by $\sin\theta$ and the other by $\cos\theta$ and adding yields the equations

$$\begin{aligned} \dot{r} &= ar(1 - r^2) - 2abr\cos^2\theta && = f \\ \dot{\theta} &= -1 + 2ab\sin\theta\cos\theta && = g \end{aligned}$$

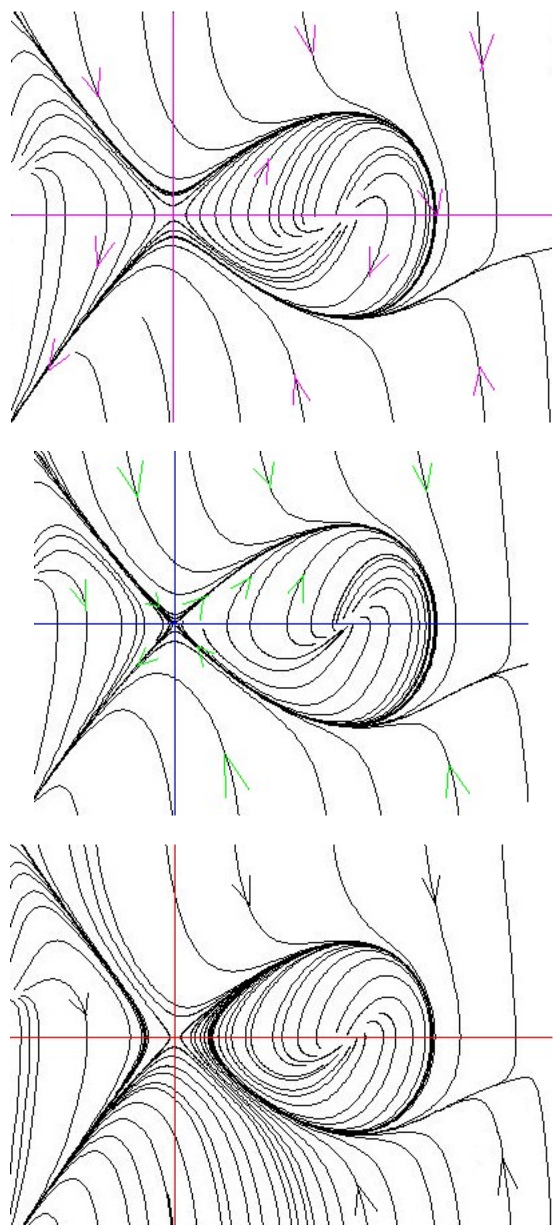


Figure 9: Homoclinic Bifurcation for $\mu = .02., 0, -.02$ plotted using 3D-XplorMath©.

Note that the second equation does not involve r so may be integrated along any closed orbit to find the period since $0 \le ab < 1$ so $\dot\theta < 0$

$$T(a,b) = \int_{\theta=\pi}^{-\pi} \frac{dt}{d\theta}\, d\theta = -\int_{\theta=-\pi}^{\pi} \frac{dt}{d\theta}\, d\theta = \int_{\theta=-\pi}^{\pi} \frac{d\theta}{1-2ab\sin\theta\cos\theta}$$

We shall not integrate this, but it may be achieved using complex analysis or a $u = \tan(\theta/2)$ substitution. The same time occurs for all closed orbits.

Suppose $b = 0$. Then the r equation reduces to

$$\dot r = ar(1-r^2)$$

which has an unstable fixed point at $r = 0$ and a stable one at $r = 1$ since $0 < a$. Thus all trajectories that don't start at the fixed points limit to the single $r = 1$ orbit. Thus $r = 1$ is the unique closed orbit. Because $0 < a$ and $1 \le 2b < 1$, then f satisfies

$$ar[1-2b-r^2] \le f = ar[1-r^2-2b\cos^2\theta] \le ar[1-r^2]$$

Let $R_1^2 = 1-2b > 0$ and $R_2 = 1$. By the left inequality, we see that $0 = aR_1[1-2b-R_1^2] \le f$ and by the right inequality $f \le aR_2[1-R_2^2] = 0$. It follows that the annulus $R_1 \le r \le R_2$ is a trapping region. The flow on the inner circle is not inward and the flow on the outer circle is not outward. Also there are no rest points because $\dot\theta < 0$ on this annulus. Thus we may apply the Poincaré Bendixson theorem: any trajectory starting in the annulus must limit to a limit cycle C in the annulus. This limit cycle is a nontrivial periodic trajectory.

13. *Consider P. Waltman's predator-prey system where $x \ge 0$ is the prey population whose growth in the absence of the predator is limited by a unit carrying capacity so it has negative growth rate for $x > 1$. The predator $y \ge 0$ dies out when no predator is present and $0 < \mu < 1$ is a parameter. Show that the system undergoes a Hopf bifurcation for some parameter value $\mu = \mu_c$ to be determined. Determine if the bifurcation is supercritical or subcritical. Show that there are no closed orbits at $\mu = \mu_c$ and on one side of μ_c. Show that there exists a closed nontrivial periodic obit on the other side. [Problem from R. Clark Robinson, An Introduction to Dynamical Systems Continuous and Discrete, 2004, Pearson Education Inc., pp. 221–224.]*

$$\begin{aligned} \dot x &= x\left(1 - x - \frac{2y}{1+2x}\right) \\ \dot y &= y\left(\frac{2x}{1+2x} - \mu\right) \end{aligned}$$

The fixed points occur at $y = 0$ when $x = 0$ or $x = 1$. If $y \ne 0$ then $x^* = \mu/2(1-\mu)$ and $(1-x)(1+2x) = 2y$ so $y^* = (2-3\mu)/4(1-\mu)^2$. This being positive implies $\mu < 2/3$. The Jacobian is

$$J(x,y) = \begin{pmatrix} 1 - 2x - \dfrac{2y}{(1+2x)^2} & -\dfrac{2x}{1+2x} \\ \dfrac{2y}{(1+2x)^2} & \dfrac{2x}{1+2x} - \mu \end{pmatrix}$$

At the the rest point $(0,0)$

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -\mu \end{pmatrix}$$

the system is an saddle. At $(1,0)$

$$J(1,0) = \begin{pmatrix} -1 & -\dfrac{2}{3} \\ 0 & \dfrac{2}{3} - \mu \end{pmatrix}$$

the system has a saddle since $\mu < 2/3$. At (x^*, y^*),

$$J\left(\frac{\mu}{2(1-\mu)}, \frac{2-3\mu}{4(1-\mu)^2}\right) = \begin{pmatrix} \dfrac{\mu(1-3\mu)}{2(1-\mu)} & -\mu \\ 1 - \frac{3}{2}\mu & 0 \end{pmatrix}$$

The determinant $\Delta = \mu(1 - \frac{3}{2}\mu) > 0$ because $\mu < 2/3$. The trace is

$$\tau = \frac{\mu(1-3\mu)}{2(1-\mu)}$$

which is positive if $\mu < \mu_c = 1/3$. In this case, the rest point (x^*, y^*) is a repeller. If $\mu \geq \mu_c$ this rest point is a global attractor and well show there are no nontrivial periodic orbits. When $\mu = \mu_c = 1/3$ we have $x_c = 1/4$, $y_c = 9/16$, $\Delta_c = 1/6$ and $\tau_c = 0$ so that the eigenvalues of $J(x_c, y_x)$ are $\pm\sqrt{\Delta_c}$. The real part of the eigenvalues are $\Re\mathrm{e}\,\tau/2$ which strictly decrease as τ increases through μ_c. Thus (x^*, y^*) is a stable for $\mu > \mu_c$. We shall show that it is also stable for $\mu = \mu_c = 1/3$, so that a supercritical Hopf bifurcation occurs at μ_c.

It turns out that we may use Dulac's Criterion at $\mu = \mu_c$. Consider the multiplier

$$h(x,y) = \left(\frac{1+2x}{2x}\right) y^{\alpha-1}$$

with $\alpha = 1/2(1-\mu)$. Then at any point in the first quadrant

$$\begin{aligned}
\operatorname{div}(hf, hg) &= \\
&= \frac{\partial}{\partial x}\left[\left(\frac{1+2x}{2x}\right) y^{\alpha-1} x \left(1 - x - \frac{2y}{1+2x}\right)\right] + \frac{\partial}{\partial y}\left[\left(\frac{1+2x}{2x}\right) y^{\alpha}\left(\frac{2x}{1+2x} - \mu\right)\right] \\
&= \frac{\partial}{\partial x}\left[\left(\frac{(1+2x)(1-x)}{2} - y\right) y^{\alpha-1}\right] + \frac{\partial}{\partial y}\left[\left(1 - \frac{(1+2x)\mu}{2x}\right) y^{\alpha}\right] \\
&= \left(\frac{1}{2} - 2x\right) y^{\alpha-1} + \alpha\left(1 - \frac{(1+2x)\mu}{2x}\right) y^{\alpha-1} \\
&= \frac{y^{\alpha-1}}{2x}\Big(x - 4x^2 + \alpha(2x - (1+2x)\mu)\Big) \\
&= \frac{y^{\alpha-1}}{2x}\Big(-4x^2 + [1 + 2\alpha(1-\mu)]x - \alpha\mu\Big) \\
&= \frac{y^{\alpha-1}}{2x}\left(-4x^2 + 2x - \frac{\mu}{2(1-\mu)}\right) \\
&= \frac{y^{\alpha-1}}{2x}\left(-\left[2x - \frac{1}{2}\right]^2 + \frac{1-3\mu}{4(1-\mu)}\right)
\end{aligned}$$

The parenthesis is nonpositive for $\mu \geq 1/3$ and possibly zero only on the line $x = 1/4$. Thus there are no periodic trajectories in the first quadrant for these μ since the integrals of the divergence over the region inside such trajectories wouldn't be zero as they must be by the divergence theorem.

To argue that there is a nontrivial closed orbit for $\mu < \mu_c = 1/3$ we use the Poincaré Bendixson Theorem. Since (x^*, y^*) is unstable then, there is a tiny ellipse $\mathcal{E}$ about (x^*, y^*) through which the flow is outgoing. We claim that a trapping region is a pentagon $\mathcal{P}$ bounded by the curves $y = 1/(2\mu)$, $x = 0$, $y = 0$, $x = 1$ and $x + y = 1/2\mu + x^*$. For $y = 1/2\mu$ and $0 \le x \le x^*$ we have $\dot{y} \le 0$ since $2x/(1 + 2x) \le \mu$. Since $\dot{x} = 0$ when $x = 0$ and $\dot{y} = 0$ when $y = 0$, no flow passes through the axes. For $x = 1$ we have $\dot{x} \le 0$ so flow enters $\mathcal{P}$ along this side. Finally flow is incoming through the fifth side since

$$\dot{x} + \dot{y} = x - x^2 - \mu y = (1 + \mu)x - x^2 - \mu\left(\frac{1}{2\mu} + x^*\right) \le \frac{(1 + \mu)^2}{4} - \frac{1}{2} \le \frac{4}{9} - \frac{1}{2} < 0$$

where we have used $ax - x^2 \le a^2/4$, $0 < \mu < 1/3$ and $0 < x^* \le 1/4$. We have shown that the annular region $\mathcal{P} - \mathcal{E}$ is a trapping region because all flow is either inward or along the boundary. Its only rest points are at $(0, 0)$ and $(0, 1)$ on the boundary. Both of these rest points have their stable manifolds along the axes. We cannot apply Poincaré Bendixson yet, because there should be no rest points on the boundary for the Theorem to apply.

Let us now describe a smaller trapping region without boundary rest points. Start at a point $y_0 = 1/2\mu$ and $0 < x_0 < x^*$. Its trajectory γ goes southwest following near the y-axis until it crosses the parabola $2y = (1 - x)(1 + 2x)$ vertically and then southeast and comes close to the origin for x_0 small enough by continuous dependence, gets turned along the positive x-axis since the origin is a saddle and follows along the x axis until it crosses $x = x^*$. This trajectory avoids the axes since different trajectories can't touch. From that point on it travels northeast so must encounter the paraFbola $2y = (1 - x)(1 + 2x)$ again which it crosses upward. From that point on it travels northwest. It can't cross $x + y = 1/2\mu + x^*$ nor $y = 1/2\mu$. Therefore it recrosses $x = x^*$ from right to left at a point below $y = 1/2\mu$ at time t_1. Define a region $\mathcal{R}$ bounded by γ from $t = 0$ to $t = t_1$, the segment from $\gamma(t_1)$ to $y = 1/2\mu$ along the line $x = x^*$, and the segment along $y = 1/2\mu$ from $x = x^*$ to $x = x_0$. Then remove the ellipse $\mathcal{E}$ to complete $\mathcal{R}$. There are no rest points along this new closed region $\mathcal{R}$. Applying the Poincaré Bendixson Theorem tells us that any trajectory starting inside $\mathcal{R}$ tends to a limit cycle, which is a closed trajectory inside $\mathcal{R}$.

14. *Consider a pendulum driven by constant torque and damped by air resistance were $\alpha, F > 0$ are dimensionless parameters in the nondimensionalized equation. Find and classify the rest points in the (θ, ν) phase plane. If you find a center in the linearization about a rest point, determine whether it is a center for the nonlinear equations, or a stable/unstable spiral. For $F > 1$ show that the system has a nontrivial limit cycle. (regard the phase plane as a cylinder). then prove that this limit cycle is unique. regarding t as a function of θ, find the differential equation for $uu = \frac{1}{2}\nu^4$. Solve this equation in the regime $\nu > 0$ andf find an exact formula for the limit cycle when $F > 1$. Decreasing F while keeping α fixed, show that the limit cycle undergoes a homoclinic bifurcation at some critical value $F_c(\alpha)$, and give an exact formula for the bifurcation curve. [Strogatz problem 8.5.5]*

$$\ddot{\theta} + \alpha\dot{\theta}|\dot{\theta}| + \sin\theta = F$$

Writing this as a system.

$$\dot{\theta} = \nu$$
$$\dot{\nu} = F - \sin\theta - \alpha\nu|\nu|$$

Rest points have $\nu = 0$ so $\sin\theta = F$. This has two roots $0 < \theta_- \le \pi/2 \le \theta_+ < \pi$ when

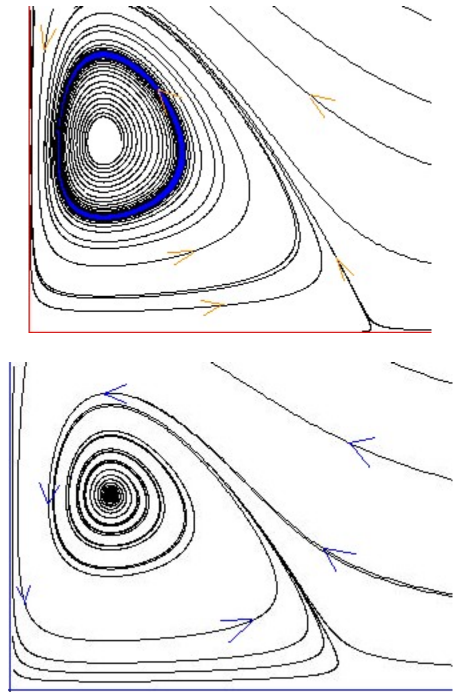


Figure 10: 3D-XplorMath© plots: $\mu = .3$ blue limit cycle, $\mu = .367$ stable rest point.

$0 < F \leq 1$, and none for $F > 1$. The Jacobian is

$$J = \begin{pmatrix} 0 & 1 \\ -\cos\theta & -2\alpha|\nu| \end{pmatrix}$$

The trace is 0 at the rest points and the determinant is $\cos\theta_{\pm}$. Thus $(\theta_+, 0)$ is a saddle and $(\theta_-, 0)$ is a center in the linearization.

To see the nonlinear nature at $(\theta_-, 0)$, consider the Liapunov Function

$$V(\theta, \nu) = \frac{1}{2}\nu^2 - \cos\theta - F\theta$$

Note that the partial derivatives $V_\theta(\theta_-, 0) = \sin\theta_- - F = 0$ and $V_{\theta\theta}(\theta_-, 0) = \cos\theta_- > 0$ so that V has a strict local minimum at $(\theta_-, 0)$. Its derivative with respect to time is

$$\dot{V} = \nu\dot{\nu} + (\sin\theta - F)\dot{\theta} = \nu(F - \sin\theta - \alpha\nu|\nu|) + (\sin\theta - F)\nu = -\alpha|\nu|^3 \leq 0$$

Noting that when $\nu = 0$ then $\dot{\nu} \neq 0$ unless $\theta = \theta_-$ so V is strictly decreasing in the neighborhood of $(\theta_0, 0)$ for flows starting away from rest points. Thus the nonlinear behavior is a stable spiral in the neighborhood of $(\theta_-, 0)$.

In case of $F > 1$ we show that the system has a periodic solution. Let us show that the Poincaré Map has a fixed point, which implies that there is a periodic solution. Denote the solution curve with initial conditions $(0, \nu_0)$ by $(\theta(t, \nu_0), \nu(t, \nu_0))$. We shall show that if ν_0 is in an interval $[\nu_1, \nu_2]$, then the solution extends to $\theta(T(\nu_0); \nu_0) = 2\pi$ where $T(\nu_0) > 0$ is the smallest time that the solution crosses $\theta = 2\pi$. Then the Poincaré Map is defined by $P(\nu_0) = \nu(T(\nu_0), \nu_0)$, the ν-value at the first crossing point.

We first construct a trapping region $0 < \nu_1 \le \nu \le \nu_2$. Since $F - 1 > 0$ there is $\nu_1 > 0$ so small that $F - 1 > \alpha\nu_1^2$. Then if $\nu = \nu_1$,

$$\dot\nu = F - \sin\theta - \alpha\nu|\nu| \ge F - 1 - \alpha\nu^2 \ge F - 1 - \alpha\nu_1^2 > 0.$$

Thus all trajectories may cross $\nu = \nu_1$ from below but cannot cross from above. Similarly, let $\nu_2 > \nu_1$ be so large that $F + 1 < \alpha\nu_1^2$. Now if $\nu = \nu_2$

$$\dot\nu = F - \sin\theta - \alpha\nu|\nu| \le F + 1 - \alpha\nu^2 = F - 1 - \alpha\nu_2^2 < 0.$$

thus flow across the upper line $\nu = \nu_2$ is strictly downward. Thus we have a trapping region. It implies that if $\nu_0 \in [\nu_1, \nu_2]$ that $\nu_2 \ge \dot\theta(t, \nu_0) = \nu(t, \nu_0) \ge \nu_1 > 0$ so that $\theta(t)$ is increasing at a uniformly positive but bounded rate. Also $\nu(t, \nu_0)$ remains bounded. This means that the solution exists long enough to exit the box $[\nu_1, \nu_2] \times [0, 2\pi]$ through the right boundary at a finite time $T(\nu_0)$. Since the equation vector field depends smoothly on (ν, θ), by continuous dependence of ODE's, $T(\nu)$ and $(\theta(t; \nu), \nu(t; \nu))$ are continuous for all possible $(\theta, \nu) \in \cup_{\nu\in[\nu_1,\nu_2]}\{\nu\} \times [0, T(\nu)]$. Thus the Poincaré Map is a continuous function that maps $[\nu_1, \nu_2]$ to itself. By the fixed point theorem, there is a point $nu^* \in [\nu_1, \nu_2]$ such that $P(\nu^*) = \nu^*$. But this implies that the trajectory $\nu(t, \nu^*)$ is 2π-periodic and $(\theta(t, \nu^*), \nu(t, \nu^*))$ is a closed trajectory on the cylinder.

To see that the limit cycle is unique, observe that there is no rest point in $\nu \ne 0$ because $\dot\theta = \nu$ doesn't vanish and for $\nu = 0$, $\dot\theta = F - \sin\theta > 0$ doesn't vanish either. It follows that there are no nontrivial closed trajectories in the (θ, ν) plane becuase such would have to enclose a rest point by index theory. The only other option is if the trajectory wraps around the cylinder. Because flow is upward through $\nu = 0$, no wrapping closed trajectory may cross the ν-axis since because it cannot return to $\nu < 0$ to close up. And no closed wrapping cycle exists in $\nu < 0$ because $\dot\nu > 0$ there so $\nu(t, \nu_0)$ cannot be periodic. Thus every closed wrapping trajectory satisfies $\nu > 0$. To argue uniqueness, suppose there are two closed wrapping trajectories $\nu(t)$ and $\tilde\nu(t)$. Because $\dot\nu > 0$, we may think of both curves as functions of θ. Let us use the energy method in the text. Consider $E = \frac{1}{2}\nu^2 - \cos\theta$. Then

$$\frac{d\nu}{d\theta} = \frac{\dot\nu}{\dot\theta} = \frac{F - \sin\theta - \alpha\nu|\nu|}{v} \tag{5}$$

The energy returns after one circuit so integrating on a closed trajectory,

$$0 = \Delta E = \int_0^{2\pi} \frac{dE}{d\theta}\, d\theta = \int_0^{2\pi} \nu\frac{d\nu}{d\theta} + \sin\theta\, d\theta = \int_0^{2\pi} F - \alpha\nu^2\, d\theta$$

So for every closed trajectory

$$\frac{2\pi F}{\alpha} = \int_0^{\pi} \nu^2\, d\theta.$$

Because two trajectories can't cross, one is above the other, say $\nu(\theta) < \tilde\nu(\theta)$. But this implies that the integral is different for the two trajectories, so both can't equal the left side constant.

Now let us assume that $F > 1$ and find the solution explicitly. In the region $\nu > 0$ we have $\dot\theta = \nu > 0$ so that by the Implicit Function Theorem, we may parameterize by $\nu(\theta)$. Let $u = \frac{1}{2}\nu^2$. We have from (5)

$$\frac{du}{d\theta} = \nu\frac{d\nu}{d\theta} = F - \sin\theta - \alpha\nu|\nu| = \dot\nu = \ddot\theta$$

Thus the quadratically damped pendulum equation becomes for $\nu > 0$

$$\frac{du}{d\theta} + 2\alpha u = F - \sin\theta$$

Multiplying by an integrating factor,

$$\frac{d}{d\theta}\left(e^{2\alpha\theta}u\right) = e^{2\alpha\theta}\left(\frac{du}{d\theta} + 2\alpha u\right) = Fe^{2\alpha\theta} - e^{2\alpha\theta}\sin\theta$$

Hence

$$e^{2\alpha\theta}u(\theta) - u(0) = \frac{F}{2\alpha}\left(e^{2\alpha\theta} - 1\right) + \frac{e^{2\alpha\theta}\left(-\cos\theta + 2\alpha\sin\theta\right) + 1}{1 + 4\alpha^2}$$

so

$$u(\theta) = u(0)e^{-2\alpha\theta} + \frac{F}{2\alpha}\left(1 - e^{-2\alpha\theta}\right) + \frac{2\alpha\sin\theta - \cos\theta + e^{-2\alpha\theta}}{1 + 4\alpha^2}$$

The limiting cycle is what remains when the transients are dropped, namely

$$u^*(\theta) = \frac{F}{2\alpha} + \frac{2\alpha\sin\theta - \cos\theta}{1 + 4\alpha^2} = \frac{F}{2\alpha} + \frac{\sin(\theta - \psi)}{\sqrt{1 + 4\alpha^2}}$$

Fixing α and letting F decrease still gives the right explicit solution as long as $\nu > 0$. This breaks down when F cant keep $u > 0$. Let $\psi(\alpha)$ be the phase angle

$$(\cos\psi, \sin\psi) = \left(\frac{2\alpha}{\sqrt{1 + 4\alpha^2}}, \frac{1}{\sqrt{1 + 4\alpha^2}}\right)$$

in other words $\tan\psi = 2\alpha$ so $\psi = \operatorname{Atn} 2\alpha$. The critical value is

$$F_c(\alpha) = \frac{2\alpha}{\sqrt{1 + 4\alpha^2}}$$

In the range $F_c < F < 1$, the system is bistable because the stable periodic cycle and the stable rest point coexist. The upward unstable manifold of the saddle does not spiral into the stable rest point, but rather asymptotes to the periodic cycle. As F decreases to $F_c(\alpha)$, the periodic cycle u^* merges with the homoclinic orbit from the saddle $(\theta_+, 0)$ that connects its unstable manifold to its stable manifold and the system loses its periodic cycle. As F continues to decrease, the upward unstable manifold of the saddle now spirals into the stable rest point instead of tending to a limit cycle. Thus a homoclinic bifurcation occurs.

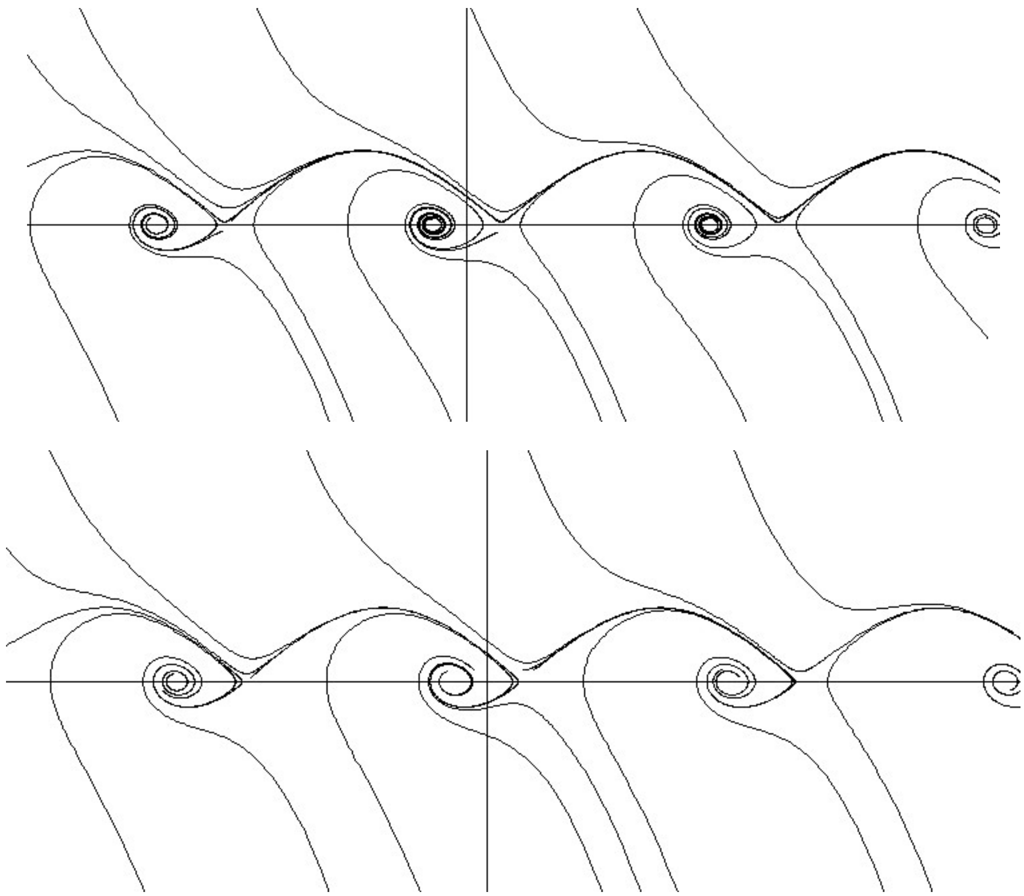


Figure 11: 3D-XplorMath© plots: $.709 = F_c < F < 1$ and $.70F < F_c$, $\alpha = .5$.

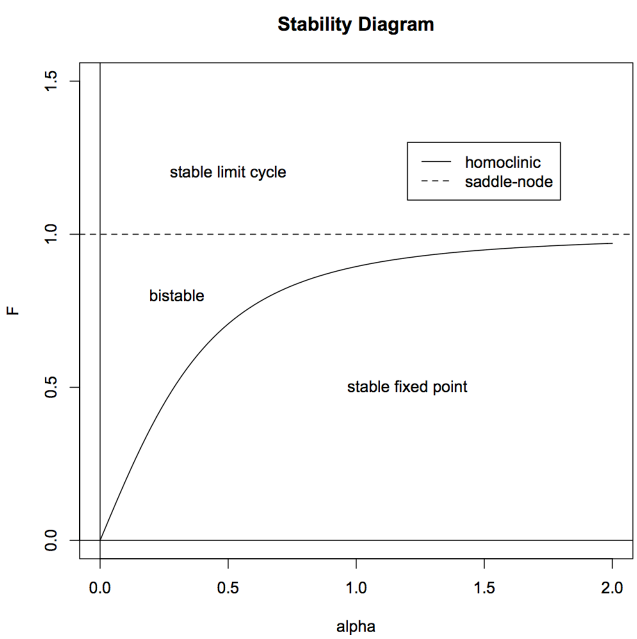


Figure 12: **R**©plot of stability diagram.