Math 5470 § 1. **Second Midterm Exam** Name: Solutions
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1. *Consider the differential equation. Find the general solution. Find the Poincaré map for 2π periodic solutions. Is there a 2π-periodic solution? Why?* $\dot{x} = (1 + \sin t)x + 2.$

   This is a linear equation. Multiply by the integrating factor

$$\frac{d}{dt}\left[e^{-t+\cos t}x\right] = e^{-t+\cos t}[\dot{x} - (1 + \sin t)x] = 2e^{-t+\cos t}$$

   Integrating with $x(0) = x_0$ where x_0 is any real gives

$$e^{-t+\cos t}x(t) - ex_0 = 2\int_0^t e^{-s+\cos s}\,ds$$

   so the general solution is

$$x(t) = e^{t+1-\cos t}x_0 + 2e^{t-\cos t}\int_0^t e^{-s+\cos s}\,ds.$$

   The Poincaré map determines where an initial point x_0 evolves under the flow in one period. Here

$$\wp(x_0) = x(2\pi) = e^{2\pi}x_0 + 2e^{2\pi-1}\int_0^{2\pi} e^{-s+\cos s}\,ds.$$

   A 2π periodic solution returns to its starting point, or satisfies the fixed point equation $x^* = \wp(x^*)$. In this case, solving we find the fixed point to be

$$x^* = \frac{2e^{2\pi-1}}{1 - e^{2\pi}}\int_0^{2\pi} e^{-s+\cos s}\,ds.$$

   Thus the solution through the fixed point is a 2π periodic solution.

   One notes that $\wp(x_0)$ is a linear function of x_0 of slope $e^{2\pi}$ so it crosses the line $y = x$ at exactly one point. Hence the periodic solution is unique. Moreover $\wp'(x^*) = e^{2\pi} > 1$, so that the periodic solution through x^* is unstable.

2. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) Statement: *A periodic orbit of the predator prey system $\dot{x} = x(1 - y)$, $\dot{y} = y(x - 1)$ is orbitally asymtotically stable.*

   False.

   All trajectories of the predator prey system are periodic since there is a conserved quantity $G(x, y) = x - \ln x + y - \ln y$. T rajectories are level sets which are concentric ovals. Thus none of the periodic orbits is attractive since nearby orbits don't approach them. Thus the orbits are not orbitally attractive. However, they are orbitally Liapunoff stable.

   (b) Statement: *Consider the $\mathcal{C}^2$ planar system $\dot{z} = f(z)$ with rest point $f(z^*) = 0$. If all eigenvalues of the Jacobian $df(z^*)$ satisfy $\lambda_i \le 0$ then z^* is Liapunov stable.*

   False.

   The asymptotic stability of a rest point of the nonlinear system can only be deduced if $\Re\mathrm{e}\,\lambda_i < 0$ for all eigenvalues of the Jacobian $df(z^*)$. Without extra conditions on f (such as reversibility) the condition $\lambda_i \le 0$ is inconclusive. For example the origin is unstable for $\dot{x} = x^3$, $\dot{y} = y^3$, but the Jacobian vanishes. It is false even for linear system $\dot{x} = y$, $\dot{y} = 0$.

(c) STATEMENT: *There are no periodic orbits of the system* $\dot{x} = -4x^3 - 2xy^2$, $\dot{y} = -2x^2y - 12y^3$.

TRUE.

Many ways to see it. It is a gradient system $\dot{z} = -\nabla V(z)$ where $V(x,y) = x^4 + x^2y^2 + 3y^4$ so all trajectories head to the origin. Or by Dulac's Criterion (Bendixson's Negative Criterion) with $g(x,y) = 1$, since $\operatorname{div} g(x,y) f(x,y) = -14x^2 - 38y^2$ which is negative except at the origin, there are no closed orbits.

3. *Find the values of the parameter a where bifurcations occur. Describe the nature of the bifurcations. Sketch the phase plane for various a's.*

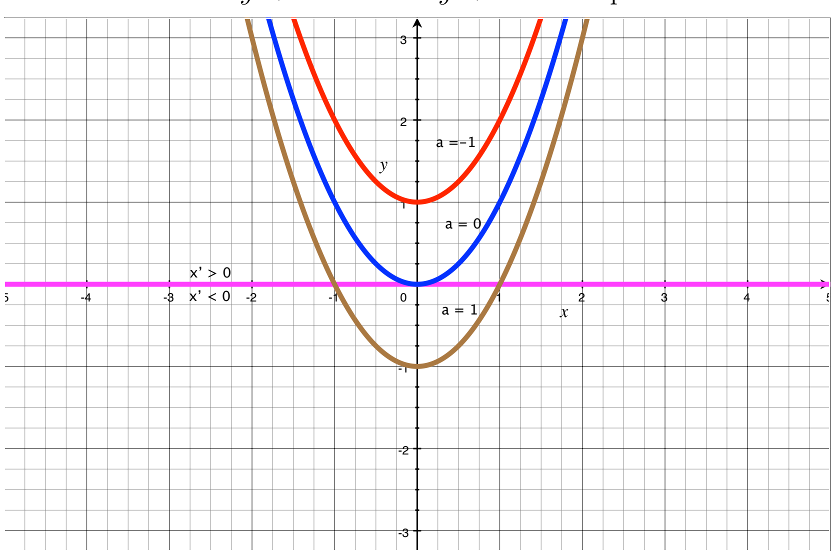
$$\begin{cases} \dot{x} = y \\ \dot{y} = x^2 - y - a \end{cases}$$

The critical points are where $0 = \dot{x} = y$ and $0 = \dot{y} = x^2 - y - a = x^2 - a$. Thus there are no rest points when $a < 0$ and $(-\sqrt{a}, 0)$ and $(+\sqrt{a}, 0)$ when $a \ge 0$. Thus the bifurcatiobn occurs at $a = 0$ and is of saddle/node type. This is seen by computing the linearizations at the rest points. The Jacobian is

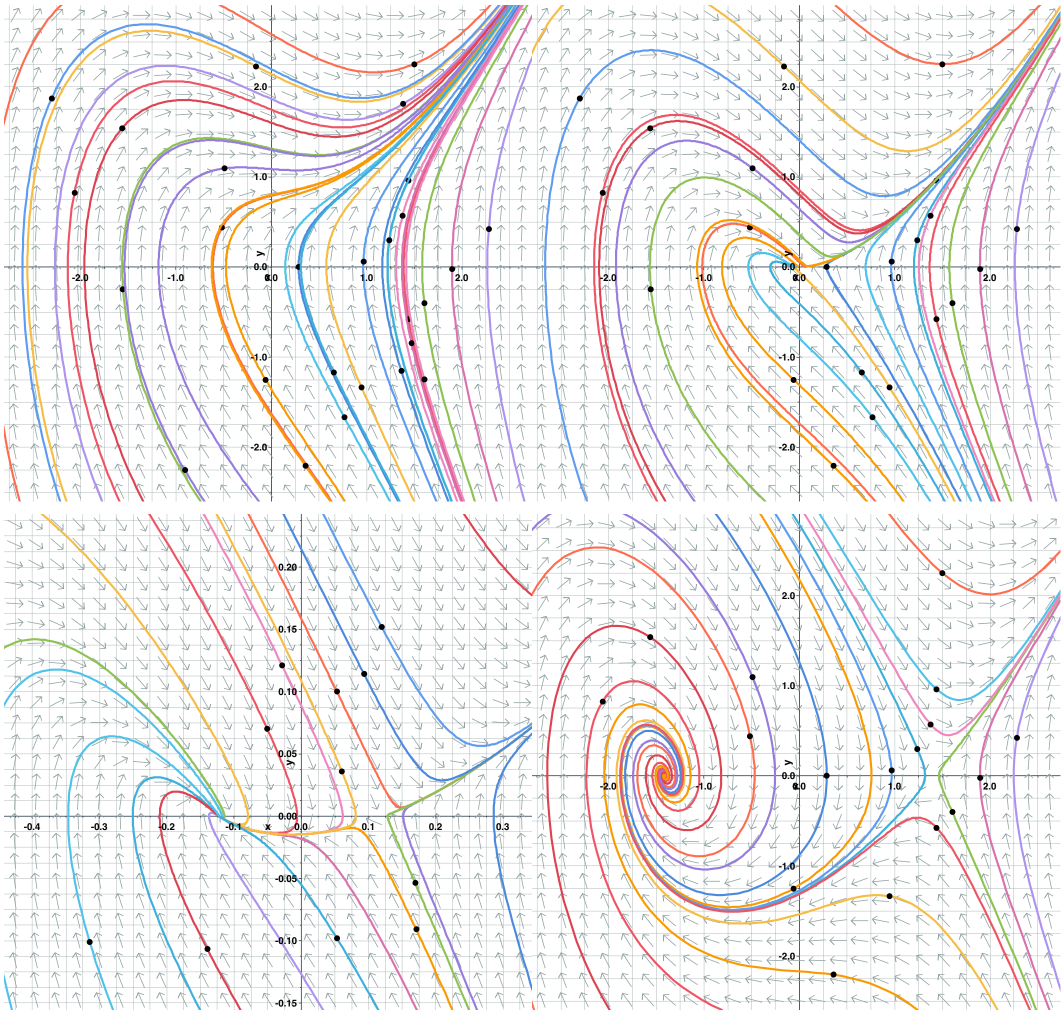
$$J(x,y) = \begin{pmatrix} 0 & 1 \\ 2x & -1 \end{pmatrix}, \qquad J(\pm\sqrt{a}, 0) = \begin{pmatrix} 0 & 1 \\ \pm 2\sqrt{a} & -1 \end{pmatrix} m$$

At $(-\sqrt{a}, 0)$ the determinant is $\Delta = 2\sqrt{a} > 0$ and trace is $\tau = -1 < 0$. In fact $\tau^2 - 4\Delta = 1 - 8\sqrt{a}$ is positive for $0 < a < 1/64$ and negative for $a > 1/64$. Thus this rest point is a stable node for $0 < a < 1/64$ and a is a stable spiral for all $a > 1/64$. The rest point persists and remains stable as a increases through $1/64$ so this is not considered a bifurcation point, even though the rest point switches from node to spiral. These are the same in the sense of topological conjugatcy. At the other rest point $(\sqrt{a}, 0)$ the determinant is $\tau^2 - 4\Delta < 0$ so that this is a saddle node for all $a > 0$.

The plots of isoclines for $a = -1, 0, 1$. Rest points at the intersections of the parabola and horizontal line with $\dot{y} < 0$ above and $\dot{y} <$ below the parablas.



The figures are “Slopes” plots for $a = -1, 0, .015, 1$.



4. *Prove that there is a nontrivial periodic solution”*

$$\begin{cases} \dot{x} = y \\ \dot{y} - x + (1 - x^2 - 2y^2)y \end{cases}$$

The rest points are at $0 = \dot{x} = y$ and $0 = \dot{y} = -x + (1 - x^2 - 2y^2)y = -z$. so $(0,0)$ is the only rest point. Let us show that there is an invariant annulus $R_1 \le x^2 + y^2 \le R_2$. Let $\rho = x^2 + y^2$. Then

$$\begin{aligned} \dot{\rho} &= 2x\dot{x} + 2y\dot{y} \\ &= 2xy + 2y(-x + (1 - x^2 - 2y^2)y) \\ &= 2(1 - x^2 - 2y^2)y^2 \end{aligned}$$

Hence

$$2(1 - 2x^2 - 2y^2)y^2 \le \dot{\rho} = 2(1 - x^2 - 2y^2)y^2 \le 2(1 - x^2 - y^2)y^2$$

Thus $\dot{\rho} \ge 0$ if $x^2 + y^2 = \frac{1}{2}$ and $\dot{\rho} \le 0$ if $x^2 + y^2 = 1$. Thus $A = \{\frac{1}{2} \le x^2 + y^2 \le 1\}$ is a closed forward invariant annulus without rest points. By the Poincaré Bendixson Theorem there is nontrivial limit cycle (periodic orbit) in A.

5. (a) *What bifurcation occurs in the equation as the parameter μ varies and why?*

$$\begin{cases} \dot{x} = y \\ \dot{y} - x + (x^2 + y^2 - \mu)y \end{cases}$$

$(0,0)$ is the only rest point for all μ. One observes that the circle $x^2 + y^2 = \mu$ is an invariant cycle for $\mu > 0$.
Putting $\rho = x^2 + y^2$, since

$$\dot{\rho} = 2x\dot{x} + 2y\dot{y} = xy - yx + (x^2 + y^2 - \mu)y^2 = (\rho - \mu)y^2$$

for ρ we have that $\rho = 0$ is a single stable rest point for $\mu \le 0$ and $\rho = 0$ an unstable and $\rho = \mu$ a stable rest points when $\mu > 0$. Thus this system undergoes a subcritical Hopf Bifurcation as μ increases through zero.

(b) *Two fireflies flash according to the equations. Do they synchronize? and if so, at what frequency? Derive any formulas you use.*

$$\begin{cases} \dot{\theta}_1 = 3 + \sin(\theta_2 - \theta_1) \\ \dot{\theta}_2 = 1 + 3\sin(\theta_1 - \theta_2) \end{cases}$$

Put $\varphi = \theta_1 - \theta_2$. Then

$$\dot{\vartheta} = 3 - 1 - (3 + 1)\sin\varphi = 2 - 4\sin\varphi = f(\varphi)$$

$f(\varphi)$ has a zeros at

$$\frac{1}{2} = \sin\varphi \qquad \text{so } \varphi^* = \frac{\pi}{6} \text{ or } \varphi^{**} = \frac{5\pi}{6}.$$

φ^* is stable since $f(\varphi) > 0$ for $\varphi^{**} - 2\pi < \varphi < \varphi^*$ and $f(\varphi) < 0$ for $\varphi^* < \varphi < \varphi^{**}$. Thus the fireflys synchronize since $\varphi \to \varphi^*$ as $t \to \infty$.
Thus the compromise frequency is the lmiting frequency of θ_1 (or θ_2) where

$$\omega^* = \frac{d}{dt}\theta_1^* = 3 + \sin(\theta_2^* - \theta_1^*) = 3 - \sin\varphi^* = 3 - \frac{1}{2} = \frac{5}{2}.$$