Homework for Math 6170 §1, Spring 2012

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Our main text this semester is Isaac Chavel, *Riemannian Geometry: A Modern Introduction, 2nd. ed.*, Cambridge, 2006. Please read the relevant sections in the text as well as any cited reference. Each problem is due three class days after its assignment, or on May 2, whichever comes first.

1. [Jan. 19.] **Induced Riemannian Metric of a Nonparametric Hypersurface.**

   Suppose that a hypersurface M^d immersed in Euclidean space $\mathbb{E}^{d+1}$ is given locally as a graph of a smooth function f defined over an open set U in $\mathbb{R}^d$.

   $$X(u_1, u_2, \ldots, u_d) = \big(u_1, u_2, \ldots, u_d, f(u_1, u_2, \ldots, u_d)\big)$$

   Define the metric in these coordinates by

   $$g_{ij}(u_1, \ldots, u_d) = \left(\frac{d}{du_i} X(u_1, \ldots, u_d)\right) \bullet \left(\frac{d}{du_j} X(u_1, \ldots, u_d)\right)$$

   (a) Let $c : [a, b] \to M$ be a piecewise $\mathcal{C}^1$ curve in M. Show that its length as a curve in Euclidean space agrees with its length computed using the metric

   $$\int_a^b |c'(t)| \, dt = \int_c ds$$

   where

   $$ds = \left(g_{ij} \frac{du_i}{dt} \frac{du_j}{dt}\right)^{\frac{1}{2}} dt$$

   and $c(t) = X(u_1(t), \ldots, u_d(t))$.

   (b) Let K in U be a compact region with smooth boundary. Show that the d-dimensional area of $X(K)$ computed as the area of a graph agrees with the area computed intrinsically using the area form:

   $$\int_K \sqrt{1 + \left(\frac{df}{du_1}\right)^2 + \cdots + \left(\frac{df}{du_d}\right)^2} \, du_1 \cdots du_d = \int_{X(K)} dA$$

   where

   $$dA = \sqrt{\det(g_{ij})} \, du_1 \wedge \ldots \wedge du_d.$$

2. [Jan. 24.] **Left Invariant Connection on a Lie Group.**

   Let G be a Lie group. (G is both a smooth differentiable manifold and a group whose multiplications and inverses are smooth operations.) Let G_e be the tangent space at the identity. For a fixed h in G, the left translation given by $\mathcal{L}_h(k) = hk$ is a diffeomorphism

because $(\mathcal{L}_h)^{-1} = \mathcal{L}_{h^{-1}}$. A vector field X on G is left invariant if for all k in G, $(d\mathcal{L}_h X)(k) = X(hk)$.

Let $\mathfrak{g}$ denote the left invariant vector fields on G. $\mathfrak{g}$ is isomorphic to G_e. It is a fact that if X, Y are in $\mathfrak{g}$ then so is their bracket $[X,Y] = XY - YX$. Hence $\mathfrak{g}$ is a Lie-subalgebra of the Lie-algebra of all vector fields on G. A one parameter subgroup of G is a Lie-group morphism $F : \mathbb{R} \to G$ of the additive Lie-group of the real numbers $\mathbb{R}$ into G such that dF is nonzero. Hence $F(s+t) = F(s)F(t)$, $F(0+t) = F(0)F(t)$ so $F(0) = e$.

Define a connection $\mathcal{D}$ by the condition $\mathcal{D}_X Y = 0$ everywhere on G for all X and left invariant Y. Since every vector field is a linear combination of left invariant vector fields with function coefficients, this condition determines the connection completely.

Show that in G with the left invariant connection, the integral curves of the left invariant vectors are the auto-parallel curves with respect to $\mathcal{D}$ such that their maximal domain of definition is the whole real line. Show also that the nonconstant auto-parallel curves of $\mathcal{D}$ starting at e can be characterized as the one-parameter subgroups of G.

3. [Jan. 31.] **Cartan's Lemma.**

Prove:

Cartan's Lemma *Let M^n be a smooth manifold and $\alpha_1, \ldots, \alpha_k, \omega_1, \ldots \omega_k$, $1 \le k \le n$, be smooth one-forms on M. Suppose that $\omega_1, \ldots \omega_k$ are independent on M and that*

$$\sum_{i=1}^{k} \alpha_i \wedge \omega_i = 0$$

for all $x \in M$. Then there are smooth functions h_{ij}, $i, j = 1, \ldots, k$, such that

$$h_{ij} = h_{ji} \qquad and \qquad \alpha_i = \sum_{j=1}^{k} h_{ij}\, \omega_j$$

for all $i = 1, \ldots, k$ and all $x \in M$.

4. [Feb. 2.] **Connection with Given Torsion.**

Suppose that $T(X,Y)$ is a skew symmetric tensor of type (1,2) on the smooth manifold M. Show that there is a unique metric compatible connection $\mathcal{D}$ on M whose torsion is T. Instead of formula I.5.4 of Chavel one gets

$$2\langle \mathcal{D}_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle + \langle Z, [X,Y]\rangle + \langle Y, [Z,X]\rangle$$
$$- \langle X, [Y,Z]\rangle + \langle Z, T(X,Y)\rangle + \langle Y, T(Z,X)\rangle - \langle X, T(Y,Z)\rangle.$$

Suppose that T satisfies the additional condition $\langle T(X,Y), Y\rangle = 0$ for all vector fields X, Y. Then the autoparallel curves for $\mathcal{D}$ coincide with the autoparallel curves of the Levi-Civita connection.

[From Gromoll, Klingenberg & Meyer, Riemannsche Geometrie im Grossen, 2nd ed., p. 87.]

5. [Feb. 7.] **Curvature of a Given Metric.**

Find the Levi-Civita connection, the Riemannian curvature and the sectional curvature of the following metrics:

(a) Weaver's metric (also known as Chebychev Coordinates of a surface.) Let $\phi(u, v)$ be a smooth function in $\mathbb{R}^2$ that satisfies

$$0 < \phi(u, v) < \pi.$$

The metric
$$ds^2 = du^2 + 2\cos(\phi)\,du\,dv + dv^2$$
is known as the weaver's metric since length is preserved along the coordinate lines which correspond to the warp and weft threads, and ϕ is the angle between the threads.

(b) Stereographic Coordinates for the Sphere. The metric, defined on the on chart $\mathbb{R}^n$, is given by
$$ds^2 = \frac{4}{(1+|x|^2)^2}\left(dx_1^2 + dx_2^2 + \cdots + dx_n^2\right)$$
where
$$|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2.$$

6. [Feb. 9.] **Equidistant Curves to a Given Curve.**

Let $\alpha : [0,a] \times (-\epsilon, \epsilon) \to M$ be a smooth map to a Riemannian manifold. Suppose that for all fixed $t_0 \in (-\epsilon, \epsilon)$, the curve $s \mapsto \alpha(s, t_0)$ for $s \in [0,a]$ is auto-parallel with unit speed and is orthogonal to the curve $t \mapsto \alpha(0,t)$ at the point $\alpha(0, t_0)$. Prove that for all (s_0, t_0) in $[0,a] \times (-\epsilon, \epsilon)$, the curves $s \mapsto f(s, t_0)$, and $t \mapsto \alpha(s_0, t)$ are orthogonal.

[do Carmo, p. 84]

7. [Feb. 14.] **First Bianchi Identity.**

(a) Let $\{w^i\}$ a local orthonormal coframe for a smooth Riemannian manifold M^n and $\{w_i{}^j\}$ its connection one forms. The Riemann curvature tensor is given by
$$dw_i{}^j - w_i{}^k \wedge w_k{}^j = \Omega_i{}^j = -\frac{1}{2}R_i{}^j{}_{pq}w^p \wedge w^q.$$

By differentiating the first structure equations,
$$dw^i = w^j \wedge w_j{}^i,$$

show the First Bianchi Identity:
$$R_i{}^j{}_{k\ell} + R_k{}^j{}_{\ell i} + R_\ell{}^j{}_{ik} = 0 \qquad \text{for all } i,\,j,\,k,\,\ell. \tag{1}$$

(b) Using the known symmetries
$$R_i{}^j{}_{k\ell} = -R_j{}^i{}_{k\ell} = -R_i{}^j{}_{\ell k} \qquad \text{for all } i,\,j,\,k,\,\ell, \tag{2}$$

and (1), show that there is another:
$$R_i{}^j{}_{k\ell} = R_k{}^\ell{}_{ij} \qquad \text{for all } i,\,j,\,k,\,\ell. \tag{3}$$

The symmetries (2), (1) are known to generate all the symmetries for curvature at a point.

(c) If the manifold is n-dimensional, let $N(n)$ be the number (dimension) of independent components of the curvature tensor $R_i{}^j{}_{pq}(x)$ that can occur at a point. Find $N(n)$. (e.g., $N(2) = 1$, $N(3) = 6$, $N(4) = 20,\dots$) [Christoffel, 1869.]

8. [Feb. 16.] **Geodesic Rays.** [Chavel, p. 37.]

Let (M, g) be a complete, noncompact, connected Riemannian manifold and let p be a point of M. Show that there is a geodesic ray emanating from p. That is, there is a unit speed curve $c : [0, \infty) \to M$ such that $c(0) = p$ and for all $t > 0$,
$$d(p, c(t)) = t.$$

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9. [Feb. 21.] **Radial Curves Minimize.** [Sakai 78[II.5.i,iii].] Let (r, θ) be ordinary polar coordinates in $\mathbb{R}^2$. Suppose the metric g satisfies

$$g\left(\frac{d}{dr}, \frac{d}{dr}\right) = 1,$$

$$g\left(\frac{d}{dr}, \frac{d}{d\theta}\right) = 0,$$

$$g\left(\frac{d}{d\theta}, \frac{d}{d\theta}\right) = f(r)^2,$$

where $f \in C^2$, $f(r, \theta) > 0$ for $r > 0$, $f(0, \theta) = 0$, $\frac{d}{dr} f(0, \theta) = 1$.

(a) Show that the radial curves $t \to (t, \theta)$ are minimizing.

(c) Find the Gauss Curvature K.

(d) Let $c \in \mathbb{R}$ be constant. Find such f so that $K = c$ for all (r, θ).

10. [Feb. 23.] **Complete Submanifold.** [Lee, p. 113.]

Let (M, g) be a complete Riemannian manifold and $N \subset M$ be a closed, embedded submanifold with the induced Riemannian metric. Show that N is complete.

11. [Feb. 28] **Divergence, Gradient and all that.** [Chavel 36[I.4], 41[I.10]]

Let f be a smooth function on the Riemannian manifold M^n.

(a) The gradient vector field grad f is defined by

$$Xf = \langle \text{grad } f, X \rangle, \qquad \text{for all } X.$$

Thus

$$\text{grad } f = \sum_{i=1}^{n} f_i e_i$$

where $\{e_i\}$ is a local orthonormal frame for M and $df = f_i w^i$ and where $\{w^i\}$ is the corresponding orthonormal coframe. Suppose that $|\text{grad } f| = 1$ everywhere. Show that the integral curves of grad f are auto-parallel.

(b) The Hessian of f is defined as the two form gotten by covariant differentiation of the one form df

$$(\text{Hess } f)(X, Y) := \nabla(df)(X, Y).$$

Thus

$$\text{Hess } f = f_{ij} w^i \otimes w^j$$

where $df_i - f_j w_i{}^j = f_{ij} w^j$. Show that the Hessian is positive semidefinite at a local minimum of f.

(c) Let X be a vector field. The divergence is the function defined by

$$\text{div } X = \text{trace}(Y \mapsto \nabla_Y X) = \sum_{i=1}^{n} \langle \nabla_{e_i} X, e_i \rangle.$$

The Laplacian is defined

$$\Delta f = \text{div grad } f = \sum_{i=1}^{n} f_{ii}.$$

Show

$$\text{div}(fX) = \langle \text{grad } f, X \rangle + f \text{ div } X,$$

and

$$\Delta(fh) = f \Delta h + 2\langle \text{grad } f, \text{grad } h \rangle + h \Delta f.$$

12. [Mar. 1] **Totally Geodesic Submanifolds.** [Chavel 96[II.3]]

Let M^m be an embedded submanifold of $\overline{M}^n$ with the induced Riemannian metric. M^m is called *totally geodesic* in $\overline{M}$ if for any geodesic γ of $\overline{M}$ for which there is a t_0 such that $\gamma(t_0) \in M$ and $\dot\gamma(t_0) \in M_{\gamma(t_0)}$, there exists an $\epsilon > 0$ such that $\gamma(t) \in M$ for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$.

(a) Show that M is totally geodesic if and only if the second fundamental form vanishes identically on M.

(b) Show that if M is a Riemannian manifold that possesses an isometry $\varphi : M \to M$, then any connected component of the set of all points left fixed by φ is a totally geodesic submanifold.

13. [Mar. 6] **Structure Constants.** [Chavel 50[I.18, I.19]]

Let G be any Lie Group with independent left invariant vector fields $\{e_1, \ldots, e_n\}$, $n = \dim G$, and dual 1-forms $\{w^1, \ldots, w^n\}$.

(a) Show that there exist constants $C_{jk}{}^i$ such that

$$[e_j, e_k] = \sum_{i=1}^n C_{jk}{}^i e_i;$$

$$dw^i = -\frac{1}{2} \sum_{j,k=1}^n C_{jk}{}^i w^j \wedge w^k;$$

$$\sum_{m=1}^n C_{jk}{}^m C_{mi}{}^r + C_{ij}{}^m C_{mk}{}^r + C_{ki}{}^m C_{mj}{}^r = 0.$$

(b) Assume G possesses a bi-invariant Riemannian metric relative to which $\{e_1, \ldots, e_n\}$ are orthonormal. Show

$$C_{ij}{}^k + C_{ik}{}^j = 0.$$

Show also that the connection and curvature forms relative to the frame $\{e_1, \ldots, e_n\}$ are given by

$$w_i{}^j = -\frac{1}{2} \sum_{k=1}^n C_{ki}{}^j w^k;$$

$$\Omega_i{}^j = \frac{1}{4} \sum_{p,q,r=1}^n C_{iq}{}^r C_{rp}{}^j w^p \wedge w^q.$$

So for left-invariant vector fields X, Y and Z on G we have

$$\langle [Z, X], Y \rangle + \langle X, [Z, Y] \rangle = 0;$$

$$\nabla_X Y = \frac{1}{2}[X, Y];$$

$$R(X, Y)Z = \frac{1}{4}\Big\{ [[Z, Y], X] - [[Z, X], Y] \Big\};$$

$$\langle R(X, Y)X, Y \rangle = \frac{1}{4}|[X, Y]|^2.$$

14. [Mar. 8] **Curvature of a Complete Metric on the Plane.** [doCarmo, p. 207.]

Suppose that g is a smooth complete metric on $\mathbb{R}^2$. Let $K(x, y)$ denote its Gauss Curvature at the point $(x, y) \in \mathbb{R}^2$. Prove that

$$\lim_{r \to \infty} \left(\inf_{x^2 + y^2 \geq r^2} K(x, y) \right) \leq 0.$$

15. [Mar. 20] **Compare Conjugate Distances.** [Chavel, p. 107.]

Let M_1 and M_2 be Riemannian manifolds of the same dimension and let $\gamma_i : [0, \ell] \to M_i$ be unit speed autoparallel curves. If $K_i(p, \pi)$ denotes the sectional curvature of at the point $p \in M_i$ for the two plane $\pi \subset (M_i)_p$, suppose that

$$\sup_\pi K_1(\gamma_1(t), \pi) \leq \inf_{\pi'} K_2(\gamma_2(t), \pi')$$

for all $t \in [0, \ell]$. Then the first conjugate point to $\gamma_1(0)$ along γ_1 cannot occur before the first conjugate point of $\gamma_2(0)$ along γ_2.

16. [Mar. 22] **Equality of Isometries.** [Chavel, p. 102.]

Suppose that M^n and N^n are Riemannian Manifolds (where M^n is connected) and that $\varphi, \psi : M \to N$, are local isometries. Assumne that at some point $p \in M^n$ we have

$$\varphi(p) = \psi(p), \qquad \text{and} \qquad d\varphi_p = d\psi_p.$$

. Show that $\phi = \psi$.

17. [Mar. 27] **Eight Models of the Hyperbolic Plane.**

Prove that the following models of the hyperbolic plane are isometric by constructing maps that exhibit the isometry. (The maps may be realized geometrically!) Show that the geodesics for each model are as stated.

(a) Upper halfplane model (X, g).

$$X = \left\{ (x, y) \in R^2 : y > 0 \right\}$$
$$g = \frac{dx^2 + dy^2}{y^2}$$

(The geodesics turn out to be vertical lines and Euclidean semicircles centered on the x-axis.)

(b) Poincaré model (X, g).

$$X = \left\{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1 \right\}$$
$$g = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y_2)^2} = \frac{dr^2 + r^2\, d\theta^2}{(1 - r^2)^2}$$

(The geodesics turn out to be lines through the origin and segments of Euclidean circles that meet $x^2 + y^2 = 1$ perpendicularly.)

(c) Klein model (X, g).

$$X = \left\{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1 \right\}$$
$$g = \frac{(1 - y^2)\, dx^2 + 2xy\, dx\, dy + (1 - x^2)\, dy^2}{(1 - x^2 - y^2)^2} = \frac{dr^2}{(1 - r^2)^2} + \frac{r^2\, d\theta^2}{1 - r^2}$$

(The geodesics turn out to be Euclidean line segments that meet X.)

(d) Hyperboloid model (X, g).

$$X = \left\{ (x, y, z) \in \mathbb{R}^3 : z = \sqrt{1 + x^2 + y^2} \right\}$$

$g =$ restriction of Minkowski metric $ds^2 = dx^2 + dy^2 - dz^2$ to X.

(The geodesics turn out to be the intersections of X with planes through the origin.)

(e) Hemisphere model (X, g).

$$X = \left\{ (x, y, z) \in R^3 : z = \sqrt{1 - x^2 - y^2} \right\}$$

$g =$ restriction of upper halfspace meric $ds^2 = \dfrac{dx^2 + dy^2 + dz^2}{z^2}$ to X.

(The geodesics turn out to be semicircles in vertical planes that meet X.)

(f) Geodesic Polar Coordinates (X, g).

$$X = \left\{ (r, \theta) \in \mathbb{R}^2 : r > 0, \ 0 < \theta < 2\pi \right\}$$
$$g = dr^2 + \sinh^2 r \, d\theta^2$$

(g) Fermi Coordinates (X, g).

$$X = \mathbb{R}^2$$
$$g = du^2 + \cosh^2 u \, dv^2.$$

(h) Horocircle Coordinates (X, g).

$$X = \mathbb{R}^2$$
$$g = du^2 + e^{2u} \, dv^2.$$

[For a wonderful treatment see J. Cannon, W. Floyd, R. Kenyon, W. Parry "Hyperbolic Geometry," in *Flavors of Geometry*, Silvio Levy, ed., MSRI Publications, **31**, 59-116, Cambridge University Press, Cambridge 1997.]

18. [Mar. 29] **Locally Symmetric Spaces.** [Chavel, p. 154.]

A Riemannian manifold (M^n, g) is called *locally symmetric* if the curvature is covariant constant $\nabla R = 0$ on all of M.

(a) Show (M^n, g) is locally symmetric if and only if for every geodesic γ on M and every vector fields X_1, X_2, X_3, X_4 that are parallel along γ we have

$$\langle R(X_1, X_2)X_3, X_4 \rangle = \text{const.}$$

along γ.

(b) Let (M^n, g) be a locally symmetric Riemannian manifold and γ a unit speed geodesic on M. Then there are constants $\kappa_2, \ldots, \kappa_n$ and parallel orthonormal frame $\{E_i\}$ with $E_1 = \dot\gamma$ along γ so that the the Jacobi fields $\mathcal{J}^\perp$ along γ are spanned by the fields

$$\left\{ \mathbf{C}_{\kappa_i}(t) E_i(t), \mathbf{S}_{\kappa_i}(t) E_i(t) \right\}_{i=2,\ldots,n},$$

where $\mathbf{C}_\kappa(t)$ and $\mathbf{S}_\kappa(t)$ generalize cosine and sine (see p. 79).

(c) Let M_1 and M_2 be locally symmetric spaces of the same dimension. Suppose we have points $p_i \in M_i$ and $\delta > 0$ so that the exponential maps $\exp_{p_1}$ and $\exp_{p_2}$ are diffeomorphisms on the δ-balls in their respective tangent spaces. Suppose that we have a linear isometry

$$\iota : (M_1)_{p_1} \to (M_2)_{p_2}$$

such that

$$\iota R(X, Y)Z = R(\iota X, \iota Y)\iota Z$$

for all $X, Y, Z \in (M_1)_{p_1}$. Show that

$$\phi = \exp_{p_2} \circ \iota \circ \exp_{p_1}^{-1} : B(p_1, \delta) \to B(p_2, \delta)$$

is an isometry.

19. [Apr. 3] **Comparing Lengths in Surfaces.** [Chavel, p. 108.]

Suppose M_1 and M_2 be two-dimensional Riemannian manifolds and K_1 and K_2 their respective Gauss curvature functions. Assume that

$$\sup_{M_1} K_1 \leq \inf_{M_2} K_2.$$

Suppose we choose points $p_i \in M_i$ and a curve $\zeta : [a, b] \to \mathbb{R}^2$. Let $\iota_i : \mathbb{R}^2 \to (M_i)_{p_i}$ be linear isometries. Let $\zeta_i = \iota_i \circ \zeta$ be the corresponding curves in the tangent spaces and $\omega_i = \exp_{p_i} \circ \zeta_i$ curves on the surfaces. Assume also that for every $\epsilon \in [a, b]$, there is no conjugate point in $(0, |\zeta(\epsilon)|]$ to p_2 along the unit speed geodesic of M_2 given by

$$t \mapsto \exp_{p_2} \left(\frac{t\zeta_2(\epsilon)}{|\zeta_2(\epsilon)|} \right).$$

Show that then there is an inequality of lengths

$$\ell(\omega_1) \geq \ell(\omega_2).$$

20. [Apr. 5] **Cut Locus.**

For all flat torii T^2 and $p \in T^2$, find the cut locus of p in the tangent space and on T^2.

Flat torii $T_\Gamma^2 = \mathbb{E}^2 / \Gamma$ are determined by the deck transformations, the subgroup Γ of the isometry group of $\mathbb{E}^2$ isomorphic to $\mathbb{Z} \times \mathbb{Z}$ given by

$$\Gamma = \{z_1 \mathbf{a}_1 + z_2 \mathbf{a}_2 : z_i \in \mathbb{Z}\}$$

where $\mathbf{a}_1$ and $\mathbf{a}_2$ are independent vectors. Two flat torii T_Γ^2 and $T_{\Gamma'}^2$ are isometric if there is an isometry ϕ of $\mathbb{E}^2$ such that $\Gamma' = \phi(\Gamma)$. Thus, up to homothety and isometry, we may suppose that $\mathbf{a}_1 = (1, 0)$ and $\mathbf{a}_2 \in \mathcal{M}$ where

$$\mathcal{M} = \left\{ (x, y) \in \mathbb{R}^2 : 0 \leq x \leq \frac{1}{2} \text{ and } y \geq \sqrt{1 - x^2} \right\}.$$

21. [Apr. 5] **Nonnegative Ricci Volume Growth Rigidity.**

[Petersen, *Riemannian Geometry*, Springer 1997, p. 270.]

Suppose that M^n is a complete Riemannian Manifold with nonnegative Ricci Curvature, $p \in M$ and satisfies

$$\lim_{r \to \infty} \frac{\text{Vol}(B(p, r))}{\omega_n r^n} = 1,$$

where $\text{Vol}(B(p, r))$ is the volume of an r-ball in M about p and ω_n is the volume of the unit ball in Euclidean Space $\mathbb{E}^n$. Show that M is isometric to $\mathbb{E}^n$.

22. [Apr. 10] **Nonnegative Ricci Manifolds have Linear Growth.** [Chavel, p. 165.]

Prove the following theorem of E. Calabi and S. T. Yau. Suppose M is a complete, noncompact manifold of nonnegative Ricci curvature. Then for every $p \in M$ there is a constant $c > 0$ so that for every $r > \dfrac{1}{c}$, the volume of the ball about p satisfies

$$\text{Vol}(B(p,r)) \geq cr.$$

Show that this estimate may be sharp, such as for a Riemannian product $M^n = N^{n-1} \times \mathbb{R}$.

23. [Apr. 12] **Linear Isoperimetric Constant.**

For a Riemannian manifold M^n, a constant c is called a *linear isoperimetric constant* if

$$\text{Vol}_n(\Omega) \leq c \, \text{Vol}_{n-1}(\partial \Omega) \tag{4}$$

for all open subsets of $\Omega \subset M$ with compact closure in M and with smooth boundary. Let M^n be a complete, noncompact, simply connected Riemannian manifold whose sectional curvatures are less than of equal to $\kappa < 0$. Then

$$c = \frac{1}{(n-1)\sqrt{-\kappa}}$$

is a linear isoperimetric constant for M. Show that this constant is sharp in the sense that for fixed κ, there are examples M and Ω where the difference in (4) is arbitrarily small.

24. [Apr. 17] **Volume Entropy vs. Growth of Fundamental Group.** [Sakai, p. 237.]

Let (M^n, g) be a compact Riemannian manifold and $(\tilde{M}, \tilde{g})$ be its universal cover with lifted metric and pr : $\tilde{M} \to M$ the projection. The fundamental group $\pi_1(M, x)$ can be identified with the deck transformation group G. Fix a point $\tilde{x} \in \tilde{M}$ so that $\text{pr}(\tilde{x}) = x$ and define a norm of $a \in G$ by $\|a\| = d(\tilde{x}, a\tilde{x})$. Using the generating set $S = \{a \in G : \|a\| \leq 3 \, \text{diam}(M)\}$, define a second norm on G given by the word metric $|a| =$ least number of generators of S whose product is a. Prove that

$$\text{diam}(M)|a| \leq \|a\| \leq 3 \, \text{diam}(M)|a|.$$

For positive r, let $N(r) = \sharp\{a \in G : \|a\| < r\}$. Define

$$h(G) = \liminf_{r \to \infty} \frac{\log\big(N(r)\big)}{r};$$

$$h(M) = \liminf_{r \to \infty} \frac{\log\big(\text{Vol}(B(p,r))\big)}{r}.$$

Show $h(G)$ and $h(M)$ are independent of the choice of $\tilde{x} \in \tilde{M}$. Show $h(G) \leq h(M)$. Show that if Ric $\geq 1 - n$ then $h(M) \leq n - 1$.

25. [Apr. 19] **Two Rays.** [Sakai, p. 190.]

Let M^n be a complete manifold of nonnegative sectional curvature. Let $w, c : [0, \infty)$ be unit speed autoparallel curves emanating from p. Suppose that c is a ray: $d(c(t), c(0)) = t$ for all $t > 0$. Suppose that the angle from $c'(0)$ to $w'(0)$ is strictly smaller that $\pi/2$. Show that

$$\lim_{s \to \infty} d(w(s), w(0)) = \infty.$$

26. [Apr. 24] **Law of Sines and Cosines.** [Chavel p. 103.]

Prove the law of cosines and the law of sines for the space of constant curvature κ. Show that as $\kappa \to 0$ the formulas limit to the Euclidean versions. e.g., if $\kappa = -1$ then for the triangle with vertex angles α, β, γ and opposite side lengths a. b, c there holds

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma;$$

$$\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma}.$$

[One argument: put one of the vertices at the z-axis intercept in the sphere or hyperboloid. Then rotate space.]

27. [May 2] **Converse of Toponogov Theorem.** [Petersen, p. 359.]

Show that the converse of Toponogov's Theorem is also true: suppose M^n is a Riemannian Manifold and $\kappa \in \mathbb{R}$ such that for all geodesic hinges $H(a, b, \gamma) \subset M$ such that a and b are minimizing geodesics (whose lengths are assumed to be less than $\pi/\sqrt{\kappa}$ if $\kappa > 0$) we have

$$|c| \leq |c^*|$$

where $\overline{H}(\bar{a}, \bar{b}, \gamma)$ is the hinge in the comparison space form $\mathbb{S}^2_\kappa$ and where c and c^* are the corresponding minimizing geodesics spanning the hinges. Then the sectional curvature of M satisfies $K \geq \kappa$.

28. [May 2] **Generators of the Fundamental Group.**

Let M^n be a compact Riemannian manifold whose sectional curvature satisfies $K \geq -1$ and whose diameter is bounded $\operatorname{diam}(M) \leq D$.

(a) Then the fundamental group $\pi_1(M, p)$ is generated by at most

$$N = 2\Big(3 + 2\cosh(2D)\Big)^{\frac{n}{2}}$$

generators.

(b) Show that for a surface M^2 of genus g, this number works out to be at least $5g^4$. Thus N is far from sharp.