Homework for Math 6410 §1, Fall 2010

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Our main text this semester is Jane Cronin, *Ordinary Differential Equations Introduction and Qualitative Theory, 3rd. ed.,* Chapman & Hall/CRC, Boca Raton, FL, 2008. Please read the relevant sections in the text as well as any cited reference. Each problem is due six class days after its assignment, or on Dec. 16, whichever comes first.

1. [Aug. 23.] **Compute a Phase Portrait using a Computer Algebra System.** This exercise asks you to figure out how to make a computer algebra system draw a phase portrait. For many of you this will already be familiar. See, *e.g.,* the MAPLE worksheet from today's lecture

   `http://www.math.utah.edu/~treiberg/M6412eg1.mws`

   `http://www.math.utah.edu/~treiberg/M6412eg1.pdf`

   or my lab notes from Math 2280,

   `http://www.math.utah.edu/~treiberg/M2282L4.mws`.

   Choose an autonomous system in the plane with at least two rest points such that one of the rest points is a saddle and another is a source or sink. Explain why your system satisfies this. (Everyone in class should have a different ODE.) Using your favorite computer algebra system, *e.g.,* MAPLE or MATLAB, plot the phase portrait indicating the background vector field and enough integral curves to show the topological character of the flow. You should include trajectories that indicate the stable and unstable directions at the saddles, trajectories at the all rest points including any that connect the nodes, as well as any seperatrices.

2. [Aug. 25.] **Iteration Scheme.** Show that the iteration scheme $\psi_0(t) \equiv A$,

   $$\psi_{n+1}(t) = A + Bt + \int_0^t (s-t)\psi_n(s)\,ds$$

   will converge to a solution of the problem $x'' + x = 0$, $x(0) = A$, $x'(0) = B$ for certain values of t. For what values of t is convergence assured? [From H. K. Wilson, *Ordinary Differential Equations,* Addison-Wesley, 1971, p.245.]

3. [Aug. 27.] **Gronwall's Inequality.** Suppose that u and v are nonnegative, continuous real valued functions defined on $[t_0, \infty)$. Assume that there is a constant $0 \le M < \infty$ so that

   $$u(t) \le M + \int_{t_0}^t u(s)v(s)\,ds$$

   for all $t \ge t_0$. Show that

   $$u(t) \le M \exp\left(\int_{t_0}^t v(s)\,ds\right)$$

   for all $t \ge t_0$. [Cronin, 40[12].]

4. [Aug. 30.] **The Contraction Mapping Principle.** Here is the abstract idea behind the Picard Theorem. Let $(\mathcal{V}, \|\cdot\|)$ be a Banach Space (a complete normed linear space). Let $0 < b < \infty$ and $0 < k < 1$ be constants and let $T : \mathcal{V} \to \mathcal{V}$ be a transformation. Suppose that for any $\phi, \psi \in \mathcal{V}$ if $\|\psi\| \le b$ then $\|T(\psi)\| \le b$ and if both $\|\phi\| \le b$ and $\|\psi\| \le b$ then

$$\|T(\psi) - T(\phi)\| \le k\|\psi - \phi\|,$$

i.e., T is a *contraction.* Prove that there exists an element η with $\|\eta\| \le b$ such that $\eta = T\eta$, that is, T has a fixed point. Prove that η is the unique fixed point among points that staisfy $\|\eta\| \le b$. [Coddington & Levinson, *Theory of Ordinary Differential Equations,* Krieger 1984, pp. 40-41.]

5. [Sept. 1.] **Solve a Delay-Differential Equation.** The delay differential equation involves past values of the unknown function x, and so its initial data φ must be given for all times $t \le 0$. Apply the Contraction Mapping Principle to show the local existence of a solution to the delay differential equation.

   Theorem. Let $b > 0$. Let $f \in \mathcal{C}(\mathbb{R}^3)$ be a function that satisfies a Lipschitz condition: there is $L < \infty$ such that for all $t, x_1, x_2, y_1, y_2 \in \mathbb{R}$,

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \le L(|x_1 - x_2| + |y_1 - y_2|).$$

   Let $g \in \mathcal{C}(\mathbb{R})$ such that $g(t) \le t$ for all t. Let $\varphi \in \mathcal{C}\big((-\infty, 0], \mathbb{R}\big)$ such that $|\varphi(t) - \varphi(0)| \le b$ for all $t \le 0$. Show that there is an $r > 0$ such that the initial value problem

$$\begin{cases} \dfrac{dx}{dt}(t) = f\big(t, x(t), x(g(t))\big) \\ \quad x(t) = \varphi(t) \qquad \text{for all } t \le 0. \end{cases}$$

   has a unique solution $x(t) \in \mathcal{C}\big((-\infty, r], \mathbb{R}\big) \cap \mathcal{C}^1\big((0, r), \mathbb{R}\big)$.

   [*cf.* Saaty, Modern Nonlinear Equations, Dover 1981, §5.5.]

6. [Sept. 3.] **Nagumo's Uniqueness Theorem.** Prove the uniqueness theorem of Nagumo (1926).

   Theorem. Suppose $f \in C(\mathbb{R}^2)$ such that

$$|f(t, y) - f(t, z)| \le \frac{|y - z|}{|t|}$$

   for all $t, y, z \in \mathbb{R}$ such that $t \ne 0$. Then the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = f(t, y) \\ \quad y(0) = 0 \end{cases}$$

   has a unique solution.

   Show that Nagumo's theorem implies the uniqueness statement in the Picard Theorem.

7. [Sept. 8.] **Global Well-Posedness of the Initial Value problem.** Suppose that for the value $\mu = \mu_0$, the parameter dependent initial value problem

$$\begin{aligned} x' &= f(t, x, \mu) \\ x(t_0) &= x_0. \end{aligned} \tag{1}$$

has a unique solution $x(t;t_0,x_0,\mu_0)$ whose domain contains the finite interval $[a,b]$. Assume that f is continuous and satisfies a Lipschitz condition with respect to x on an open set G in (t,x,μ) space where G contains the solution curve

$$\Big\{(t,x(t;x_0,\mu_0),\mu_0)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^q : a\le t\le b\Big\}.$$

(Alternatively, you may assume $f\in\mathcal{C}^1(G,\mathbb{R}^n)$.) Show that for a sufficiently small $r>0$, the initial value problem (1) has a unique solution $x(t;\tau,\xi,\mu)$ whose domain contains $[a,b]$ for every $(\tau,\xi,\mu)\in\mathcal{U}_r$ where

$$\mathcal{U}_r=\Big\{(t,x,\mu)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^q:\ a\le t\le b,\ |x-x(t;t_0,x_0,\mu_0)|<r \text{ and } |\mu-\mu_0|<r\Big\}.$$

Moreover, x is continuous in $(\tau,\xi,\mu)\in\mathcal{U}_r$ uniformly for $t\in[a,b]$.

[This is a slightly sharpened version of Cronin, 40[13].]

8. [Sept. 10.] **Find a Periodic Solution.** This exercise gives conditions for an ordinary differential equation to admit periodic solutions.

   (a) Let $J=[0,1]$ denote an interval and let $\phi\in C(J,J)$ be a continuous transformation. Show that ϕ admits at least one fixed point. (A fixed point is $y\in J$ so that $\phi(y)=y$.)

   (b) Assume that $f\in C(\mathbb{R}\times[-1,1])$ such that for some $\lambda<\infty$ and some $0<T<\infty$ we have

   $$\begin{gathered}|f(t,y_1)-f(x,y_2)|\le\lambda|y_1-y_2|,\\ f(T+t,y_1)=f(t,y_1),\\ f(t,-1)f(t,+1)<0\end{gathered}$$

   for all $t\in\mathbb{R}$ and all $y_1,y_2\in[-1,1]$. Using $\{a\}$, show that the equation $y'=f(t,y)$ has at least one solution periodic of period T.

   (c) Apply (b) to $y'=a(t)y+b(t)$ where a,b are T periodic functions.

9. [Sept. 13.] **Escape Times.** Show that each solution $\big(x(t),y(t)\big)$ of the initial value problem

$$\begin{cases}x'= & y+x^2\\ y'= & x+y^2\end{cases}\qquad\qquad \begin{cases}x(0)= & x_0\\ y(0)= & y_0\end{cases}$$

with $x_0>0$ and $y_0>0$ cannot exist on an interval of the form $[0,\infty)$.

[*cf.* Wilson, Ordinary Differential Equations, Addison-Wesley, 1971, p.255.]

10. [Sept. 15] **Application of Liouville's Theorem.** Find a solution of the IVP for Bessel's Equation of order zero

$$\begin{cases}\qquad x''+\dfrac{1}{t}\,x'+x=0\\ x(0)=1,\qquad x'(0)=0\end{cases}$$

by assuming the solution has a power series representation (or use Frobenius Method.) Use Liouville's formula for the Wronskian to find a differential equation for a second linearly independent solution of the differential equation. Show that this solution blows up like $\log t$ as $t\to 0$. [*cf.* Fritz John, Ordinary Differential Equations, Courant Institute of Mathematical Sciences, 1965, p. 90.]

11. [Sept. 17.] **Jordan Form.** Find the generalized eigenvectors, the Jordan form and the general solution

$$\dot{\mathbf{y}} = \begin{pmatrix} 6 & 6 & 4 \\ -2 & -2 & -4 \\ 2 & 6 & 8 \end{pmatrix} \mathbf{y}.$$

12. [Sept. 20.] **Jordan Form implies Real Canonical Form.** Let A be a real 2×2 matrix whose eigenvalues are $a \pm ib$ where $a, b \in \mathbb{R}$ and $b \neq 0$. Using the Jordan Canonical Form for complex matrices, show that there is a real matrix Q so that $Q^{-1}AQ = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

13. [Sept. 22.] **Just Multiply by t.** Consider the nth order constant coefficient linear homogeneous scalar equation

$$x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_1 x' + a_0 x = 0$$

where a_i are complex constants. Convert to a first order system $\mathbf{x}' = A\mathbf{x}$. Show that the geometric multiplicity of every eigenvalue of A is one. Show that a basis of solutions is $\{t^k \exp(\mu_i t)\}$ where $i = 1, \ldots, s$ correspond to distinct eigenvalues μ_i, and $0 \le k < m_i$ where m_i is the algebraic multiplicity of μ_i.

[*cf.* Gerald Teschl, Ordinary Differential Equations and Dynamical Systems, on-line book, Universität Wien, 2010, p. 68.]

14. [Sept. 24.] **To Use Jordan Form or Not to Use Jordan Form.** Sometimes the use of the Jordan Canonical Form and matrices with multiple eigenvalues can be avoided using the following considerations.

    (a) Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$. Show that given $\epsilon > 0$ there exists a matrix B with distinct eigenvalues so that $\|A - B\| \le \epsilon$.

    (b) Give three proofs of $\det(e^A) = e^{\operatorname{trace}(A)}$.

    (c) Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$. By a simpler algorithm than finding the Jordan Form, one can change basis by a P that transforms A to upper triangular

$$P^{-1}AP = U = \begin{pmatrix} u_{11} & u_{12} & \ldots & u_{1n} \\ 0 & u_{22} & \ldots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & u_{nn} \end{pmatrix}. \qquad (2)$$

    Show that this fact can be used instead of Jordan Form to characterize all solutions of $\dot{y} = Ay$ (as linear combinations of products of certain exponentials, polynomials and trigonometric functions). [*c.f.,* Bellman, *Stability Theory of Differential Equations*, pp. 21–25.])

    (d) Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$. Show that given $\epsilon > 0$ there exists a nonsingular P such that in addition to (2) we may arrange that $\sum_{i<j} |u_{ij}| < \epsilon$.

15. [Sept. 27.] **Variation of Parameters Formula.** Solve the inhomogeneous linear system

$$\begin{cases} \dot{\mathbf{x}} = A(t)\,\mathbf{x} + \mathbf{b}(t), \\ \quad \mathbf{x}(t_0) = \mathbf{c}; \end{cases}$$

where

$$A(t) = \begin{pmatrix} -2\cos^2 t & -1 - \sin 2t \\ 1 - \sin 2t & -2\sin^2 t \end{pmatrix}, \qquad \mathbf{b}(t) = \begin{pmatrix} 1 \\ e^{-2t} \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Hint: a fundamental matrix is given by

$$U(t,0) = \begin{pmatrix} e^{-2t}\cos t & -\sin t \\ e^{-2t}\sin t & \cos t \end{pmatrix}.$$

[*cf.* Perko, Differential Equations and Dynamical Systems, Springer, 1991, p. 62.]

16. [Sept. 29.] **Trouble Lurks Near Every Point in a Linear System.** Suppose at least one eigenvalue of the real $n \times n$ matrix A has a positive real part. Prove that for any $v \in \mathbb{R}^n$, $\varepsilon > 0$ there is a solution to $x' = Ax$ so that

$$|x(0) - v| < \varepsilon \quad \text{and} \quad \lim_{t\to\infty} |x(t)| = \infty.$$

[*cf.* Hirsch & Smale, Differential Equations, Dynamical Systems and Linear Algebra, Academic Press, 1974, p. 137.]

17. [Oct. 1.] **Periodic linear equation.** Let $A(t)$ be a continuous real matrix function. Consider the equation

$$\dot{x} = A(t)x, \qquad x \in \mathbb{R}^n, \qquad A(t+T) = A(t).$$

Let $\Phi(t)$ be the fundamental matrix with $\Phi(0) = I$.

(a) Show that there is at least one nontrivial solution $\chi(t)$ such that $\chi(t+T) = \mu\chi(t)$, where μ is an eigenvalue of $\Phi(T)$.

(b) Suppose that $\Phi(T)$ has n distinct eigenvalues μ_i, $i = 1, \ldots, n$. Show that there are n linearly independent solutions of the form $x_i = p_i(t)e^{\rho_i t}$ where $p_i(t)$ is T-periodic. How is ρ_i related to μ_i?

(c) Consider the equation $\dot{x} = f(t)\,A_0\,x$, $x \in \mathbb{R}^2$, with $f(t)$ a scalar T-periodic function and A_0 a constant matrix with real distinct eigenvalues. Determine the corresponding Floquet Multipliers.

[U. Utah PhD Preliminary Examination in Differential Equations, August 2008.]

18. [Oct. 4.] **Blowup in periodic linear equation.** Let $\phi(t)$ be real, continuous and periodic with period π. Consider the scalar equation

$$y''(t) - (\cos^2 t)\,y'(t) + \phi(t)\,y(t) = 0, \qquad t \in \mathbb{R}.$$

Show that there is a solution that goes to ∞ as $t \to \infty$. [*cf.* James H. Liu, A First Course in the Qualitative Theory of Differential Equations, Prentice Hall 2003, p. 162.]

19. [Oct. 6.] **Boundedness in Hill's Equation.** Show that if $|\epsilon|$ is small enough, then all solutions are bounded

$$\ddot{u} + [1 + \epsilon \sin(3t)]u = 0$$

[U. Utah PhD Preliminary Examination in Differential Equations, January 2004.]

20. [Oct. 8.] **Discrete Dynamical Systems.** Let $T \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}^n)$. Consider the difference equation

$$\begin{aligned} x(0) &= x, \\ x(n+1) &= T(x(n)). \end{aligned} \tag{3}$$

Writing $Tx := T(x)$, a solution sequence of (3) can be given as the n-th iterates $x(n) = T^n x$ where $T^0 = I$ is the identity function and $T^n = TT^{n-1}$. The solution automatically exists and is unique on nonnegative integers $\mathbf{Z}_+$. Solutions $T^n x$ depend continuously on x since T is continuous. The *forward orbit* of a point x is the set $\{T^n x : n = 0, 1, 2, \ldots\}$. A set $H \subset \mathbb{R}^n$ is *positively (negatively) invariant* if $T(H) \subset H$ ($H \subset T(H)$). H is said to be *invariant* if $T(H) = H$, that is if it is both positively and negatively invariant. A closed invariant set is *invariantly connected* if it is not the union of two nonempty disjoint invariant closed sets. The solution $T^n x$ starting from a given point x is *periodic* or *cyclic* if for some $k > 0$, $T^k x = x$. The least such k is called the *period* of the solution or the *order* of the cycle. If $k = 1$ then x is a *fixed point* of T or an *equilibrium state* of (3). $T_n x$ (defined for all $n \in \mathbf{Z}$) is called an *extension of the solution $T^n x$ to* $\mathbf{Z}$ if $T_0 x = x$ and $T(T_n x) = T_{n+1} x$ for all $n \in \mathbf{Z}$. Thus $T_n x = T^n x$ for $n \geq 0$.

   (a) Show that a finite set (a finite number of points) is invariantly connected if and only if it is a periodic orbit. [Hint: Any permutation can be written as a product of disjoint cycles.]

   (b) Show that a set H is invariant if and only if each motion starting in H has an extension to $\mathbf{Z}$ that is in H for all n.

   (c) Show, however, that an invariant set H may have an extension to $\mathbf{Z}$ from a point in H which is not in H.

[J. P. LaSalle in J. Hale's *Studies in ODE*, Mathematical Association of America, 1977, p. 7]

21. [Oct. 18.] **Polar Coordinates.** Consider the differential equation where a and b are positive parameters

$$\begin{aligned} \dot{x} &= -\frac{ax}{\sqrt{x^2+y^2}} \\ \dot{y} &= -\frac{ay}{\sqrt{x^2+y^2}} + b \end{aligned}$$

which models the flight of a bird heading toward the origin at constant speed a, that is moved off course by a steady wind of velocity b. Determine the conditions on a and b to ensure that a solution starting at $(p, 0)$, for $p > 0$ reaches the origin. Hint: change to polar coordinates and study the phase portrait of the differential equation on the cylinder. [Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, p. 86.]

22. [Oct. 20.] Suppose $f \in \mathcal{C}^2(\mathbb{R}^n)$. Prove that the ω-limit set of an orbit of a gradient system

$$\dot{x} = \nabla f(x)$$

consists entirely of rest points. [Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, p. 88.]

23. [Oct. 22.] **Feedback Control.** Consider the equation for the pendulum of length ℓ, mass m in a viscous medium with friction proportional to the velocity of the pendulum. Suppose that the objective is to stabilize the pendulum in the vertical position (above its pivot) by a control mechanism which can move the pendulum horizontally. Let us assume that ϑ is the angle from the vertical position measured in a clockwise direction and the restoring force v due to the control mechanism is a linear function of ϑ and $\dot\vartheta$, that is, $v(\vartheta, \dot\vartheta) = c_1\vartheta + c_2\dot\vartheta$. Explain why the differential equation describing the motion is

$$m\ddot\vartheta + k\dot\vartheta - \frac{mg}{\ell}\sin\vartheta - \frac{1}{\ell}(c_1\vartheta + c_2\dot\vartheta)\cos\vartheta = 0.$$

Show that constants c_1 and c_2 can be chosen in such a way as to make the equilibrium point $(\vartheta, \dot\vartheta) = (0, 0)$ asymptotically stable. [*cf.* J. Hale and H. Koçek, *Dynamics and Bifurcations*, Springer 1991, p. 277.]

24. [Oct. 25.] **Asymptotically Stable Equilibrium in a Discrete Dynamical System.**

(a) Let A be a complex $n \times n$ matrix such that $|\lambda| < \gamma$ for all eigenvalues λ of A. Show that there is a norm $\|\cdot\|$ on $\mathbb{C}^n$ so that $\|Ax\| \le \gamma\|x\|$ for all $x \in \mathbb{C}^n$.

(b) Let $P \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ such that $P(0) = 0$ and $|\lambda| < 1$ for all eigenvalues of $DP(0)$. Show that 0 is an asymptotically stable fixed point of the discrete dynamical system in $\mathbb{R}^n$

$$\begin{aligned} x_{n+1} &= P(x_n), \\ x_1 &= \xi. \end{aligned}$$

25. [Oct. 27.] **A Condition for Asymptotic Stability.** Suppose that the zero solution of $\dot x = Ax$ is asymptotically stable. Suppose that $g(t, x) \in \mathcal{C}^1(\mathbb{R}^{n+1}, \mathbb{R}^n)$ satisfies $g(t, 0) = 0$ and

$$|g(t, x)| \le h(t)|x|, \qquad \text{for all } t \ge 0 \text{ and } x \in \mathbb{R}^n,$$

where $h(t)$ satisfies for positive constants k and r,

$$\int_0^t h(t)\, dt \le kt + r, \qquad \text{for all } t \ge 0.$$

Show that there is a constant $k_0 = k_0(A) > 0$ such that if $k \le k_0$, then the zero solution of

$$\dot x = Ax + g(t, x)$$

is asymptotically stable. [*cf.* James H. Liu, A First Course in the Qualitative Theory of Differential Equations, Prentice Hall 2003, p. 243.]

26. [Oct. 29.] **Stability of a Periodic Orbit.** Find a periodic solution to the system

$$\begin{aligned} \dot x &= x - y - x(x^2 + y^2) \\ \dot y &= x + y - y(x^2 + y^2) \\ \dot z &= -z, \end{aligned}$$

and determine its stability type. In particular, compute the Floquet Multipliers for the fundamental matrix associated with the periodic orbit. Is it orbitally asymptotically stable? Is it asymptotically stable? [Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, p. 196.]

27. [Nov. 1.] **International Whaling Commission Model.** A simple rescaled delay difference equation for modeling the population u_n of sexually mature baleen whales is

$$u_{n+1} = su_n + R(u_{n-T}), \qquad 0 < s < 1,$$

where T is an integer corresponding to time to sexual maturity and R is the number that augments the adult population from births T years earlier. If

$$R(u) = (1-s)[1 + q(1-u)]u$$

where $q > 0$ describes fecundity increase due to low density and the delay is $T = 1$, derive the condition for a positive steady state u^* to be stable and find for which q it holds. [J. D. Murray, *Mathematical Biology,* Biomathematics Texts 19, Springer 1989, p. 62.]

28. [Nov. 3.] **Stationary Points of a Hamiltonian System.** Show that the system is Hamiltonian.

$$\dot{x} = (x^2 - 1)(3y^2 - 1)$$
$$\dot{y} = -2xy(y^2 - 1)$$

Find the equilibrium points and classify them. Find the Hamiltonian. Using obvious exact solutions and the Hamiltonian property, draw a rough sketch of the phase diagram. [D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers, 4th ed.*, Oxford U. Press, 2007, p. 79.]

29. [Nov. 5.] **Particle in a Force Field.** Consider the motion of a particle in a central field, that is, suppose

$$m\ddot{x} = -\nabla U(x), \qquad x \in \mathbb{R}^3 \backslash \{0\},$$

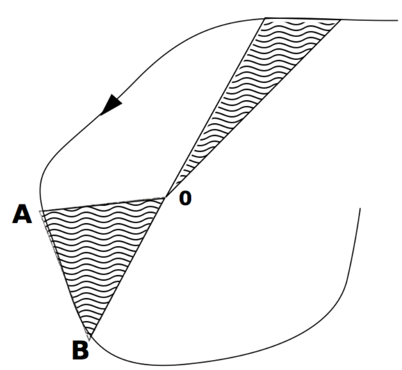
where $U(x) = U_0(|x|)$ and $U_0 \in \mathcal{C}^2((0,\infty))$.

(a) Prove that the angular momentum M relative to the point 0 is "conserved," where M is defined by the cross product

$$M := x \times m\dot{x}.$$

(b) Show that all orbits are planar (in a plane perpendicular to M).

(c) Prove Kepler's Law, which says that the radius vector "sweeps out equal area in equal time."



[H. Amann, *Ordinary Differential Equations: An Introduction to Nonlinear Analysis*, Walter de Gruyter, 1990, p. 48.]

30. [Nov. 8.] **Existence of a Periodic Orbit.** A model for an autocatalytic chemical reaction is given by the nondimensionalized Brusselator System

$$\dot{x} = 1 - 4x + x^2 y,$$
$$\dot{y} = 3x - x^2 y;$$

where $x, y \geq 0$ correspond to concentrations. Show that the trapezoidal region with vertices $(\frac{1}{4}, 0)$, $(13, 0)$, $(1, 12)$, $(\frac{1}{4}, 12)$ is a forward invariant set for this system. Show that it has a nonconstant periodic trajectory. [University of Utah Preliminary Examination in Differential Equations, Autumn 2004.]

31. [Nov. 10.] **Dulac's Criterion.** Prove the following theorem.
Theorem. Let $X \subset \mathbb{R}^2$ be an annular domain. Let $f \in \mathcal{C}^1(X, \mathbb{R}^2)$ and let $\rho \in \mathcal{C}^1(X, \mathbb{R})$. Show that if $\operatorname{div}(\rho f) \neq 0$ for all of X then the equation $x' = f(x)$ has at most one periodic solution in X.

Use this to show that the van der Pol oscillator ($\lambda = \text{const.} \neq 0$)

$$\dot{x} = y$$
$$\dot{y} = -x + \lambda(1 - x^2)y$$

has at most one limit cycle in the plane. Hint: let $\rho = (x^2 + y^2 - 1)^{-1/2}$. [Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, p. 90.]

32. [Nov. 12.] **Find a Liapunov Function or use LaSalle's Invariance Principle.** Show that the zero solution is asymptotically stable

$$\ddot{x} + (\dot{x})^3 + x = 0.$$

[Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, p. 27.]

33. [Nov. 15.] **Četaev's Theorem.** Show that the zero solution is not stable

$$\dot{x} = x^3 + xy$$
$$\dot{y} = -y + y^2 + xy - x^3.$$

[*cf.* J. Hale and H. Koçek, *Dynamics and Bifurcations*, Springer 1991, p. 286.]

34. [Nov. 17.] **Stable and Unstable Manifolds.** Find the stable manifold W^s and unstable manifold W^u near the origin of the system

$$\dot{x} = -x$$
$$\dot{y} = -y + x^2$$
$$\dot{z} = z + y^2.$$

[*cf.* Perko, Differential Equations and Dynamical Systems, Springer, 1991, p. 116–117.]

35. [Nov. 19.] **Center Manifold.** Find a center manifold for the system

$$\dot{x} = -xy$$
$$\dot{y} = -y + x^2 - 2y^2$$

through the rest point at the origin. Find a differential equation for the dynamics on the center manifold. Show that every nearby solution is attracted to the center manifold.

Hint: Look for a center manifold that is a graph $y = \psi(x)$ of the form

$$\psi(x) = \sum_{k=2}^{\infty} a_x x^k$$

using the condition for invariance $\dot{y} = \psi'(x)\dot{x}$ and $\psi'(0) = \psi(0) = 0$. Find the first few terms of the expansion, guess the answer and check. Then get the equation for the induced flow on the center manifold. [Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, p. 304.]

36. [Nov. 22.] **Hartman-Grobman Theorem.** Find a homeomorphism H in a neighborhood of 0 that establishes an isochronous flow equivalence between the flow of the differential system and the flow of the linearized system, *i.e.,* $H(\phi(t,x)) = e^{tA}H(x)$ where $A = Df(0)$ and $\phi(t, x_0)$ is the solution of $\dot{\mathbf{x}} = f(\mathbf{x})$, the nonlinear system given by

$$\begin{aligned}\dot{x} &= -x\\ \dot{y} &= -y + xz,\\ \dot{z} &= z.\end{aligned}$$

[In 8.5.10, Liu discusses the approximation used in the proof, but you can guess H from the solutions and verify.]

37. [Nov. 24.] **Continuation from Harmonic Oscillator.** Show that Rayleigh's Equation has a periodic solution for small ε parameter values that is a continuations from an $\varepsilon = 0$ solution

$$\ddot{x} + \varepsilon\left(\dot{x} - \dot{x}^3\right) + x = 0.$$

[Chicone, *e.g., Ordinary Differential Equations with Applications,* Springer 1999, pp. 318–324.]

38. [Nov. 26.] **Stability at a Non-hyperbolic Critical Point.** Show that the origin is asymptotically stable for the system

$$\begin{aligned}\dot{x} &= -y + yz + (y - x)(x^2 + y^2),\\ \dot{y} &= x - xz - (x + y)(x^2 + y^2),\\ \dot{z} &= -z + (1 - 2z)(x^2 + y^2).\end{aligned}$$

Hint: Show that the surface $z = x^2 + y^2$ is invariant. [D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers, 4th ed.*, Oxford U. Press, 2007, pp. 435.]

39. [Nov. 29.] **Persistence of a Periodic Orbit.** Let $f(t,x)$ and $F(t)$ be $\mathcal{C}^1$ functions that are T-periodic. Assume that $f(t,0) = 0$ and $D_x f(t,0) = 0$ for all t. Let A be an $n \times n$ matrix such that the equation $\dot{y} = Ay$ admits no nontrivial T-periodic solutions. Show that for sufficiently small ϵ, there is a unique T-periodic solution $x(t,\epsilon)$ of

$$\dot{x} = Ax + f(t,x) + \epsilon F(t)$$

such that $x(t,\epsilon) \to 0$ as $\epsilon \to 0$ for all t. [Coddington & Levinson, *Theory of Ordinary Differential Equations,* Krieger 1984, p. 370.]

40. [Dec. 1.] **Linstedt's Method.** Find the frequency and amplitude to first order and the frequency-amplitude relation for periodic solutions of

$$\begin{gathered}\ddot{x} + x - \varepsilon(x^3 + x^5) = 0.\\ x(0,\varepsilon) = a_0, \qquad \dot{x}(0,\varepsilon) = 0.\end{gathered}$$

[D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers, 4th ed.*, Oxford U. Press, 2007, pp. 171.]

41. [Dec. 3.] **Persistence in an Autonomous System.** Let $f, g \in \mathcal{C}^1(\mathbb{R}^2)$. Show that for small ε, periodic solutions exist close to the $\varepsilon = 0$ solution $(\cos t, \sin t)$.

$$\dot{x} = \frac{x}{\sqrt{x^2 + y^2}} - x - y + \varepsilon f(x, y),$$
$$\dot{y} = \frac{y}{\sqrt{x^2 + y^2}} - y + x + \varepsilon g(x, y),$$

[*cf.* Fritz John, Ordinary Differential Equations, Courant Institute of Mathematical Sciences, 1965, p. 148.]

42. [Dec. 6.] **Bifurcation in a Forest Model.** Consider Ludwig's model for the dynamics of a balsam fir forest infested by the spruce budworm. The condition of the forest is described by $S(t)$, the average size of trees and $E(t)$, the "energy reserve," a measure of the forests health. In the presence of a constant budworm population B, the forest dynamics is given by

$$\dot{S} = r_S S \left(1 - \frac{S}{K_S}\frac{K_E}{E}\right),$$
$$\dot{E} = r_E E \left(1 - \frac{E}{K_E}\right) - P\frac{B}{S},$$

where r_S, r_E, K_S, K_E, P are positive parameters. Nondimensionalize the system. Sketch the nullclines. Show that there are two fixed points if B is small and none if B is large. Analyze the bifurcation at the critical value of B. What kind of bifurcation is it and why? Sketch the phase portraits for both large and small B. [S. Strogatz, *Nonlinear Dynamics and Chaos*, Westview 1994, p. 285.]

43. [Dec. 8.] **Bifurcation in the Brusselator.** Show that the system undergoes a supercritical Hopf Bifurcation as the parameter passes through 2.

$$\dot{x} = 1 - (1 + \lambda)x + x^2 y.$$
$$\dot{y} = \lambda x - x^2 y.$$

[Y. Kuznetsov, *Elements of Applied Bifurcation Theory, 3rd. ed.*, Springer, 2004, p. 103.]

44. [Dec. 10.] **Feedback Stiffness Control.** Moon and Rand [1985] proposed a method of damping low modes in the vibration of trusses x by actively tensioning stiffening cables v.

$$\dot{x} = y,$$
$$\dot{y} = -x - xv,$$
$$\dot{v} = -v + \alpha x^2.$$

Show that the origin $(x, y, v) = (0, 0, 0)$ is asymtotically stable if $\alpha < 0$ and unstable if $\alpha > 0$. [Y. Kuznetsov, *Elements of Applied Bifurcation Theory, 3rd. ed.*, Springer, 2004, p. 188, `http://audiophile.tam.cornell.edu/randpdf/moon.pdf`]