Homework for Math 6410 §1, Fall 2012

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Our main text this semester is Lawrence Perko, *Differential Equations and Dynamical Systems, 3rd. ed.,* Springer, 1991. Please read the relevant sections in the text as well as any cited reference. Each problem is due three class days after its assignment, or on Monday, Dec. 10, whichever comes first.

1. [Aug. 20.] **Compute a Phase Portrait using a Computer Algebra System.** This exercise asks you to figure out how to make a computer algebra system draw a phase portrait. For many of you this will already be familiar. See, *e.g.,* the MAPLE worksheet from today's lecture

   `http://www.math.utah.edu/~treiberg/M6412eg1.mws`

   `http://www.math.utah.edu/~treiberg/M6412eg1.pdf`

   or my lab notes from Math 2280,

   `http://www.math.utah.edu/~treiberg/M2282L4.mws`.

   Choose an autonomous system in the plane with at least two rest points such that one of the rest points is a saddle and another is a source or sink. Explain why your system satisfies this. (Everyone in class should have a different ODE.) Using your favorite computer algebra system, *e.g.,* MAPLE or MATLAB, plot the phase portrait indicating the background vector field and enough integral curves to show the topological character of the flow. You should include trajectories that indicate the stable and unstable directions at the saddles, trajectories at the all rest points including any that connect the nodes, as well as any seperatrices.

2. [Aug. 22.] **Continuous Dependence for Constant Coefficient Linear Systems.** Let A be an $n \times n$ real matrix. For every $x_0 \in \mathbb{R}^n$, let $\varphi(t; x_0)$ denote the solution of the IVP

$$\begin{cases} \dfrac{dx}{dt} = Ax, \\ \\ x(0) = x_0. \end{cases}$$

   Note that $\varphi(t; x_0)$ is defined for all $t \in \mathbb{R}$. For fixed $t \in \mathbb{R}$ show that

$$\lim_{y \to x_0} \varphi(t; y) = \varphi(t, x_0).$$

3. [Aug. 24.] **Real Canonical Form.** Let A be a real 2×2 matrix whose eigenvalues are $a \pm bi$ where $a, b \in \mathbb{R}$ such that $b \neq 0$. Show that there is a real matrix Q so that

$$Q^{-1}AQ = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Use this fact to solve the system

$$\begin{aligned} x' &= -13x - 10y \\ y' &= 20x + 15y \end{aligned}$$

4. [Aug. 27.] **Jordan Form.** Find the generalized eigenvectors, the Jordan form and the general solution

$$\dot{\mathbf{y}} = \begin{pmatrix} 6 & 6 & 4 \\ -2 & -2 & -4 \\ 2 & 6 & 8 \end{pmatrix} \mathbf{y}.$$

5. [Aug. 29.] **Just Multiply by t.** Consider the n-th order constant coefficient linear homogeneous scalar equation

$$x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_1 x' + a_0 x = 0$$

where a_i are complex constants. Convert to a first order differential system $\mathbf{x}' = A\mathbf{x}$. Show that the geometric multiplicity of every eigenvalue of A is one. Show that a basis of solutions is $\{t^k \exp(\mu_i t)\}$ where $i = 1, \ldots, s$ correspond to the distinct eigenvalues μ_i and $0 \leq k < m_i$ where m_i is the algebraic multiplicity of μ_i. [*cf.,* Gerald Teschl, *Ordinary Differential Equations and Dynamical Systems,* Amer. Math. Soc., 2012, p.68.]

6. [Aug. 31.] **To Use Jordan Form or Not to Use Jordan Form.** Sometimes the use of the Jordan Canonical Form and matrices with multiple eigenvalues can be avoided using the following considerations.

   (a) Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$. Show that given $\epsilon > 0$ there exists a matrix B with distinct eigenvalues so that $\|A - B\| \leq \epsilon$.

   (b) Give three proofs of $\det(e^A) = e^{\operatorname{trace}(A)}$.

   (c) Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$. By a simpler algorithm than finding the Jordan Form, one can change basis by a P that transforms A to upper triangular

$$P^{-1}AP = U = \begin{pmatrix} u_{11} & u_{12} & \ldots & u_{1n} \\ 0 & u_{22} & \ldots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & u_{nn} \end{pmatrix}. \tag{1}$$

   Show that this fact can be used instead of Jordan Form to characterize all solutions of $\dot{y} = Ay$ (as linear combinations of products of certain exponentials, polynomials and trigonometric functions). [*c.f.,* Bellman, *Stability Theory of Differential Equations*, pp. 21–25.])

(d) Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$. Show that given $\epsilon > 0$ there exists a nonsingular P such that in addition to (1) we may arrange that $\sum_{i<j} |u_{ij}| < \epsilon$.

(e) [Optional.] Find all continuous scalar valued functions $f \in C(\mathcal{M}_{n\times n}(\mathbb{C}), \mathbb{C})$ so that

$$f(AB) = f(A)f(B) \qquad \text{for all } A, B.$$

You can probably find several different arguments on your own. [*ibid.*; or Kurosh, *Higher Algebra*, p. 334.]

7. [Sept 5.] **The Contraction Mapping Principle and a Delay Differential Equation.** The local existence follows from an abstract fixed point theorem.

(a) Let $(\mathcal{X}, \|\cdot\|)$ be a Banach Space (a complete normed linear space). Let $0 < b < \infty$ and $0 < k < 1$ be constants and let $T : \mathcal{X} \to \mathcal{X}$ be a transformation. Suppose that for any $\phi, \psi \in \mathcal{X}$ if $\|\psi\| \le b$ then $\|T(\psi)\| \le b$ and if $\|\phi\| \le b$ and $\|\psi\| \le b$ then

$$\|T(\psi) - T(\phi)\| \le k\|\psi - \phi\|,$$

i.e., T is a *cantraction.* Prove that there exists an element ζ with $\|\zeta\| \le b$ such that $\zeta = T\zeta$, that is, T has a fixed point. Prove that ζ is the unique fixed point among points that staisfy $\|\zeta\| \le b$. [*cf.* Perko 77[5]]

(b) The delay differential equation involves past values of the unknown function x, and so its initial data φ must be given for all times $t \le 0$. Using the Contraction Mapping Principle (a.), show the local existence of a solution to the delay differential equation.

Theorem. Let $b > 0$. Let $f \in \mathcal{C}(\mathbb{R}^3)$ be a function that satisfies a Lipschitz condition: there is $L < \infty$ such that for all $t, x_1, x_2, y_1, y_2 \in \mathbb{R}$,

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \le L(|x_1 - x_2| + |y_1 - y_2|).$$

Let $g \in \mathcal{C}(\mathbb{R})$ such that $g(t) \le t$ for all t. Let $\varphi \in \mathcal{C}\big((-\infty, 0], \mathbb{R}\big)$ such that $|\varphi(t)-\varphi(0)| \le b$ for all $t \le 0$. Show that there is an $r > 0$ such that the initial value problem

$$\begin{cases} \dfrac{dx}{dt}(t) = f\big(t, x(t), x(g(t))\big) \\ x(t) = \varphi(t) \qquad \text{for all } t \le 0. \end{cases}$$

has a unique solution $x(t) \in \mathcal{C}\big((-\infty, r], \mathbb{R}\big) \cap \mathcal{C}^1\big((0, r), \mathbb{R}\big)$.

[*cf.* Saaty, Modern Nonlinear Equations, Dover 1981, §5.5.]

8. [Sept. 7.] **Nagumo's Uniqueness Theorem.** Prove the theorem and show that it implies the uniqueness statement in the Picard Theorem.

Theorem. [Nagumo, 1926] Suppose $f \in C(\mathbb{R}^2)$ such that

$$|f(t, y) - f(t, z)| \le \frac{|y - z|}{|t|}$$

for all $t, y, z \in \mathbb{R}$ such that $t \ne 0$. Then the initial value problem

$$\frac{dy}{dt} = f(t, y),$$
$$y(0) = 0,$$

has a unique solution.

9. [Sept. 10.] **Existence via Schauder's Theorem.** Give another proof of the Peano Existence Theorem using the Schauder Fixed Point Theorem.

   Theorem. [Peano Existence Theorem] Let $\Omega \subset \mathbb{R} \times \mathbb{R}^n$ be a domain and $f \in \mathcal{C}(\Omega, \mathbb{R}^n)$. Then for any $(t_0, x_0) \in \Omega$ there is $a > 0$ and a continuously differentiable function $x(t) : [t_0 - a, t_0 + a] \to \mathbb{R}^n$ which solves the initial value problem

$$\begin{aligned} \frac{dx}{dt} &= f(t, (x(t))), \qquad \text{for all } t \in [t_0 - a, t_0 + a]; \\ x(t_0) &= x_0. \end{aligned}$$

   Theorem. [Schauder Fixed Point Theorem] Let $\mathcal{A}$ be a closed, bounded, convex subset of a Banach space $\mathcal{X}$ and $T : \mathcal{A} \to \mathcal{A}$ be a completely continuous function. Then T has a fixed point in $\mathcal{A}$.

   A subset $\mathcal{A}$ of a Banach space is *compact* if any sequence in $\{\phi_i\}_{i=1,2,\ldots} \subset \mathcal{A}$ has a subsequence that converges to an element in $\mathcal{A}$. f is *compact* if for any bounded set $\mathcal{A} \subset \mathcal{X}$, the closure of the set $f(\mathcal{A})$ is compact. f is *completely continuous* if it is both compact and continuous. [*cf.* Hale, p. 14.]

10. [Sept. 12.] **Compare Solutions of Two Mathieu Equations.** One solution for Problem 19 of the 2010 Math 6410 depended on comparing the solutions of the perturbed and unperturbed problems. Find a sharp estimate for the difference in values and derivatives at $T = \frac{2\pi}{3}$ of the solutions for the two initial value problems, where u_0, u_1, ϵ are constants. In fact one needed to show $|y(T; 1, 0) + \dot{y}(T; 0, 1)| < 2$ where $y(t; u_0, u_1)$ solves the IVP. Does this hold for small $|\epsilon|$?

$$\begin{aligned} \ddot{x} + x &= 0, & \ddot{y} + (1 + \epsilon \sin(3t))y &= 0, \\ x(0) &= u_0, & y(0) &= u_0, \\ \dot{x}(0) &= u_1; & \dot{y}(0) &= u_1. \end{aligned}$$

11. [Sept. 14.] **Find a Periodic Solution.** This exercise gives conditions for an ordinary differential equation to admit periodic solutions.

    (a) Let $J = [0, 1]$ denote an interval and let $\phi \in C(J, J)$ be a continuous transformation. Show that ϕ admits at least one fixed point. (A fixed point is $y \in J$ so that $\phi(y) = y$.)

    (b) Assume that $f \in C(\mathbb{R} \times [-1, 1])$ such that for some $L < \infty$ and some $0 < T < \infty$ we have

$$\begin{aligned} |f(t, y_1) - f(t, y_2)| &\le L|y_1 - y_2|, \\ f(T + t, y_1) &= f(t, y_1), \\ f(t, -1)f(t, +1) &< 0 \end{aligned}$$

    for all $t \in \mathbb{R}$ and all $y_1, y_2 \in [-1, 1]$. Using (a), show that the equation $y' = f(t, y)$ has at least one solution periodic of period T.

    (c) Apply (b) to $y' = a(t)y + b(t)$ where a, b are T periodic functions.

12. [Sept. 17.] **Concrete Variational Equation.** Let

$$f \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 + x_1^2 \\ x_3 + x_1^2 \end{pmatrix}.$$

Find the solution $\varphi(t, y) \in \mathbb{R}^3$ of

$$\frac{dx}{dt} = f\big(x(t)\big),$$
$$x(0) = y.$$

Find

$$\Phi(t, y) = D_2\, \varphi(t, y).$$

Show that it satisfies the variational equation

$$\frac{d\Phi}{dt} = Df\big(\varphi(t, y)\big) \cdot \Phi(t, y),$$
$$\Phi(0) = I.$$

[Perko, p. 84.]

13. [Sept. 19.] **Global Well-Posedness of the Initial Value Problem.** Assume that f is continuous and satisfies a local Lipschitz condition with respect to x on an open set $G \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$ and let $(t_0, x_0, \mu_0) \in G$. Suppose that for $\tau = t_0$, $\xi = x_0$ and $\mu = \mu_0$, the initial value problem

$$\begin{aligned} x' &= f(t, x, \mu), \\ x(\tau) &= \xi \end{aligned} \tag{2}$$

has a unique solution $\varphi(t; t_0, x_0, \mu_0)$ whose domain contains the interval $[a, b]$. Assume that G contains the curve

$$\{(t, \varphi(t; t_0, x_0, \mu_0), \mu_0) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m : a \le t \le b\}\,.$$

Show that for a sufficiently small $r > 0$, for every $(\tau, \xi, \mu) \in \mathcal{U}_r$ the initial value problem (2) has a unique solution $\varphi(t; \tau, \xi, \mu)$ whose domain contains $[a, b]$, where

$$\mathcal{U}_r = \{(\tau, \xi, \mu) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m : a \le \tau \le b,\ |\xi - \varphi(\tau; t_0, x_0, \mu_0)| < r \text{ and } |\mu - \mu_0| < r\}\,.$$

Moreover x is continuous for $(t, \tau, \xi, \mu) \in [a, b] \times \mathcal{U}_r$, uniformly for $t \in [a, b]$. [Cronin, 40[13].]

14. [Sept. 21.] **Variation of Parameters Formula.** Solve the inhomogeneous linear system

$$\begin{cases} \dot{\mathbf{x}} = A(t)\, \mathbf{x} + \mathbf{b}(t), \\ \quad \mathbf{x}(t_0) = \mathbf{c}; \end{cases}$$

where

$$A(t) = \begin{pmatrix} -2\cos^2 t & -1 - \sin 2t \\ 1 - \sin 2t & -2\sin^2 t \end{pmatrix}, \qquad \mathbf{b}(t) = \begin{pmatrix} 1 \\ e^{-2t} \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Hint: Show that a fundamental matrix is given by

$$U(t, 0) = \begin{pmatrix} e^{-2t}\cos t & -\sin t \\ e^{-2t}\sin t & \cos t \end{pmatrix}.$$

[*cf.* Perko, Differential Equations and Dynamical Systems, Springer, 1991, p. 62.]

15. [Sept. 24.] **Condition for Asymptotic Stability.** Suppose that the zero solution to $\dot{x} = Ax$ is asymptotically stable. Let $g(t, x) \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ satisfy $g(t, 0) = 0$ and

$$|g(t, x)| \le h(t)|x|, \qquad \text{for all } t \ge 0 \text{ and } x \in \mathbb{R}^n,$$

where $h(t)$ satisfies for positive constants k and r,

$$\int_0^t h(s)\, ds \le kt + r, \qquad \text{for all } t \ge 0.$$

Show that there is a constant $k_0(A) > 0$ such that if $k \le k_0$, then the zero solution of

$$\dot{x} = Ax + g(t, x)$$

is asymptotically stable. [*cf.* James H. Liu, *A First Course in the Qualitative Theory of Differential Equations,* Prentice Hall 2003, p. 243.]

16. [Sept. 26.] **Feedback Control.** Consider the equation for the pendulum of length ℓ, mass m in a viscous medium with friction proportional to the velocity of the pendulum. Suppose that the objective is to stabilize the pendulum in the vertical position (above its pivot) by a control mechanism which can move the pendulum horizontally. Let us assume that ϑ is the angle from the vertical position measured in a clockwise direction and the restoring force v due to the control mechanism is a linear function of ϑ and $\dot{\vartheta}$, that is, $v(\vartheta, \dot{\vartheta}) = c_1\vartheta + c_2\dot{\vartheta}$. Explain why the differential equation describing the motion is

$$m\ddot{\vartheta} + k\dot{\vartheta} - \frac{mg}{\ell}\, \sin\vartheta - \frac{1}{\ell}(c_1\vartheta + c_2\dot{\vartheta})\, \cos\vartheta = 0.$$

Show that constants c_1 and c_2 can be chosen in such a way as to make the equilibrium point $(\vartheta, \dot{\vartheta}) = (0, 0)$ asymptotically stable. [*cf.* J. Hale and H. Koçek, *Dynamics and Bifurcations*, Springer 1991, p. 277.]

17. [Sept. 28.] **Discrete Dynamical Systems.** Let $T \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}^n)$. Consider the difference equation

$$\begin{aligned} x(0) &= x, \\ x(n+1) &= T(x(n)). \end{aligned} \tag{3}$$

Writing $Tx := T(x)$, a solution sequence of (3) can be given as the n-th iterates $x(n) = T^n x$ where $T^0 = I$ is the identity function and $T^n = TT^{n-1}$. The solution automatically exists and is unique on nonnegative integers $\mathbf{Z}_+$. Solutions $T^n x$ depend continuously on x since T is continuous. The *forward orbit* of a point x is the set $\{T^n x : n = 0, 1, 2, \ldots\}$. A set $H \subset \mathbb{R}^n$ is *positively (negatively) invariant* if $T(H) \subset H$ ($H \subset T(H)$). H is said to be *invariant* if $T(H) = H$, that is if it is both positively and negatively invariant. A closed invariant set is *invariantly connected* if it is not the union of two nonempty disjoint invariant closed sets. The solution $T^n x$ starting from a given point x is *periodic* or *cyclic* if for some $k > 0$, $T^k x = x$. The least such k is called the *period* of the solution or the *order* of the cycle. If $k = 1$ then x is a *fixed point* of T or an *equilibrium state* of (3). $T_n x$ (defined for all $n \in \mathbf{Z}$) is called an *extension of the solution* $T^n x$ *to* $\mathbf{Z}$ if $T_0 x = x$ and $T(T_n x) = T_{n+1} x$ for all $n \in \mathbf{Z}$. Thus $T_n x = T^n x$ for $n \ge 0$.

   (a) Show that a finite set (a finite number of points) is invariantly connected if and only if it is a periodic orbit. [Hint: Any permutation can be written as a product of disjoint cycles.]

   (b) Show that a set H is invariant if and only if each motion starting in H has an extension to $\mathbf{Z}$ that is in H for all n.

(c) Show, however, that an invariant set H may have an extension to $\mathbf{Z}$ from a point in H which is not in H.

[J. P. LaSalle in J. Hale's *Studies in ODE*, Mathematical Association of America, 1977, p. 7]

18. [Oct. 1.] **Stable and Unstable Manifolds.** Find the stable manifold W^s and unstable manifold W^u near the origin of the system

$$\begin{aligned} \dot{\mathbf{x}} &= -x \\ \dot{y} &= -y + x^2 \\ \dot{z} &= z + y^2. \end{aligned}$$

[*cf.* Perko, Differential Equations and Dynamical Systems, Springer, 1991, p. 116–117.]

19. [Oct. 3.] **Traveling Wave.** For constant $r > 0$ let $u(x, t)$ be a real valued function satisfying Fischer's Euqation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + ru(1-u), \qquad \text{for } x \in \mathbb{R} \text{ and } t \geq 0.$$

which models the spread of disease. A special solution is the *travelling wave,* $u(t, x) = v(x - ct)$ where $c > 0$ is the wave speed.

(a) Show that the traveling wave satisfies the ODE

$$v'' + cv' + rv(1 - v) = 0,$$

where $v' = dv/ds$ and $s = x - ct$.

(b) Show that for every $c \geq 2\sqrt{r}$, Fischer's Equation has a travelling wave solution satisfying $v(s) \to 1$ as $s \to -\infty$ and $v(s) \to 0$ as $s \to \infty$ with $v'(s) < 0$ for all s. [Hint: Discuss the stability properties of equilibrium points. Use special properties of the unstable manifold at $(1, 0)$. Show there is a forward invariant triangle in the (v, v') plane bounded by the lines $v' = 0$, $v = 1$ and $v' = -\mu v$ for appropriate $\mu > 0$.]

[*cf.* J. Hale and H. Koçek, *Dynamics and Bifurcations*, Springer 1991, p. 299.]

20. [Oct. 5.] **Hartman-Grobman Theorem.** Find a homeomorphism H in a neighborhood of 0 that establishes an isochronous flow equivalence between the flow of the differential system and the flow of the linearized system, *i.e.,* $H(\phi(t, x)) = e^{tA} H(x)$ where $A = Df(0)$ and $\phi(t, x_0)$ is the solution of $\dot{\mathbf{x}} = f(\mathbf{x})$, the nonlinear system given by

$$\begin{aligned} \dot{x} &= -x \\ \dot{y} &= -y + xz, \\ \dot{z} &= z. \end{aligned}$$

[In 8.5.10, Liu discusses the approximation used in the proof, but you can guess H from the solutions and verify.]

21. [Oct. 15.] **Asymptotically Stable Equilibrium in a Discrete Dynamical System.**

(a) Let A be a real $n \times n$ matrix such that $|\lambda| < \gamma$ for all eigenvalues λ of A. Show that there is a norm $\| \cdot \|$ on $\mathbb{R}^n$ so that $\|Ax\| \leq \gamma \|x\|$ for all $x \in \mathbb{R}^n$.

(b) Let $P \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ such that $P(0) = 0$ and $|\lambda| < 1$ for all eigenvalues of $DP(0)$. Show that 0 is an asymptotically stable fixed point of the discrete dynamical system in $\mathbb{R}^n$

$$\begin{aligned} x_{n+1} &= P(x_n), \\ x_1 &= \xi. \end{aligned}$$

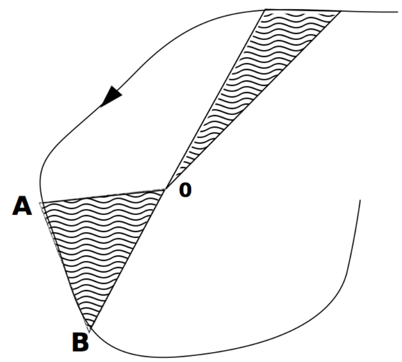
22. [Oct. 17.] **Liapunov Function.** Use a Liapunov Function to show that the zero solution is asymptotically stable
$$\ddot{x} + (2 + 3x^2)\,\dot{x} + x = 0.$$
Hint: Show that this equation is equivalent to the system
$$\begin{aligned}\dot{x} &= y - x^3\\ \dot{y} &= -x + 2x^3 - 2y.\end{aligned}$$
[D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers, 4th ed.*, Oxford U. Press, 2007, pp. 348–349.]

23. [Oct. 19.] **Četaev's Theorem.** Show that the zero solution is not asymptotically stable.
$$\begin{aligned}\dot{x} &= x^3 + xy\\ \dot{y} &= -y + y^2 + xy - x^3.\end{aligned}$$
[J. Hale and H. Koçek, *Dynamics and Bifurcations*, Springer 1991, p. 286.]

24. [Oct. 22.] **LaSalle's Invariance Principle.** Use LaSalle's Invariance Principle of find a Liapunov Function to show that the zero solution is asymptotically stable
$$\ddot{x} + (\dot{x})^3 + x = 0.$$
[Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, p. 27.]

25. [Oct. 24.] **Stationary Points of a Hamiltonian System.** Show that the system is Hamiltonian.
$$\begin{aligned}\dot{x} &= (x^2 - 1)(3y^2 - 1)\\ \dot{y} &= -2xy(y^2 - 1)\end{aligned}$$
Find the equilibrium points and classify them. Find the Hamiltonian. Using obvious exact solutions and the Hamiltonian property, draw a rough sketch of the phase diagram. [D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers, 4th ed.*, Oxford U. Press, 2007, p. 79.]

26. [Oct. 26.] **Particle in a Central Force Field.** Consider the motion of a particle om a cemntral force field. That is, suppose that
$$m\ddot{x} = -\nabla U(x), x \in \mathbb{R}^3\backslash\{0\},$$
where $U(x) = U_0(|x|)$ and $U_0 \in \mathcal{C}^2((0,\infty))$.
   (a) Prove that the angular momentum M relative to the point 0 os "conserved," where M is defined by the cross product
   $$M = x \times m\dot{x}.$$
   (b) Show that all orbits are planar (in a plane perpendicular to M).
   (c) Prove Kepler's Law, which says that the radius vector "sweeps out equal area in equal time."



[H. Amann, *Ordinary Differential Equations: An Introduction to Nonlinear Analysis,* Walter de Gruyter, 1990, p. 48.]

27. [Oct. 29.] **Brusselator System.** Show that there is a nonconstant periodic trajectory for the system

$$\begin{aligned}\dot{x} &= 1 - 4x + x^2 y\\ \dot{y} &= 3x - x^2 y\end{aligned}$$

[University of Utah Ph.D. Preliminary Examination in Differential Equations, August 2004.]

28. [Oct. 31.] **Dulac's Criterion.** Prove the following theorem.
Theorem. Let $A \subset \mathbb{R}^2$ be an annular domain. Let $f \in \mathcal{C}^1(A, \mathbb{R}^2)$ and let $\rho \in \mathcal{C}^1(A, \mathbb{R})$. Show that if $\operatorname{div}(\rho f)$ is not identically zero and does not change signs in any open subset of A then the equation $x' = f(x)$ has at most one periodic solution in A.

Use this to show that the van der Pol oscillator ($\lambda = \text{const.} \neq 0$)

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -x + \lambda(1 - x^2)y\end{aligned}$$

has at most one limit cycle in the plane. Hint: let $\rho = (x^2 + y^2 - 1)^{-1/2}$. [Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, p. 90.]

29. [Nov. 2.] ***T*-Periodic Linear Equations.** Consider the T-periodic non-autonomous linear differential equation

$$\dot{x} = A(t)\, x, \qquad x \in \mathbb{R}^n, \qquad A(t + T) = A(t).$$

Let $\Phi(t)$ be the fundamental matrix with $\Phi(0) = I$.

(a) Show that there is at least one nontrivial solution $\chi(t)$ such that $\chi(t + T) = \mu\chi(t)$, where μ is an eigenvalue of $\Phi(T)$.

(b) Suppose that $\Phi(T)$ has n distinct eigenvalues μ_i, $i = 1, \ldots, n$. Show that there are n linearly independent solutions of the form $x_i = p_i(t)e^{\rho_i t}$ where $p_i(t)$ is T-periodic. How is ρ_i related to μ_i?

(c) Consider the equation $\dot{x} = f(t)\, A_0\, x$, $x \in \mathbb{R}^2$, with $f(t)$ a scalar T-periodic function and A_0 a constant matrix with real distinct eigenvalues. Determine the corresponding Floquet Multipliers.

[U. Utah PhD Preliminary Examination in Differential Equations, August 2008.]

30. [Nov. 5.] **Blowup in a periodic linear equation.** Let $\phi(t)$ be a real, continuous, π-periodic function. Consider the scalar equation

$$\ddot{x} - (\cos^2 t)\dot{x} + \phi(t)\,x = 0.$$

Show that there is a solution that tends to infininty as $t \to \infty$. [*cf.* James H. Liu, *A First Course in the Qualitative Theory of Differential Equations,* Prentice Hall 2003, p. 162.]

31. [Nov. 7.] Show that if $|\varepsilon|$ is small enough, then all solutions are bounded.

$$\ddot{x} + \big[1 + \varepsilon \sin 3t\big]x = 0.$$

[U. Utah PhD Preliminary Examination in Differential Equations, January 2004.]

32. [Nov. 9.] **Stability of a Periodic Orbit.** Find a periodic solution to the system

$$\begin{aligned}\dot{x} &= -y + x(1 - x^2 - y^2)\\ \dot{y} &= x + y(1 - x^2 - y^2)\\ \dot{z} &= z,\end{aligned}$$

Let Σ be a halfplane whose boundary is the z axis. Determine the Poincare map $P : \Sigma \to \Sigma$ and determine its differential at the periodic orbit. Determine the stability type. In particular, compute the Floquet Multipliers for the fundamental matrix associated with the periodic orbit. Is it orbitally asymptotically stable? Is it asymptotically stable? [*cf.* Perko, Differential Equations and Dynamical Systems, Springer, 1991, p. 201.]

33. [Nov 12.] **International Whaling Commission Model.** A simple rescaled difference equation for modeling the population u_n of sexually mature baleen whales is

$$u_{n+1} = su_n + R(u_{n-T}),$$

where $0 < s < 1$ is the survival parameter, T is an integer corresponding to time to sexual maturity, and $R(u_{n-T})$ is the recruitment function that augments the adult population from births T years earlier. If

$$R(u) = (1 - s)[1 - q(1 - u)]u,$$

where $q > 0$ describes the fecundity increase due to low density and the delay is $T = 1$, derive the condition for a positive steady state u^* to be stable and find for which q it holds. [J. D. Murray, *Mathematicsl Biology,* Biomathematics Texts 19, Springer 1989, p. 62.]

34. [Nov. 14.] **Stability in a Non-Autonomous Equation.** Show the stability of the solution

$$x(t) = \sqrt{2b}\cos\frac{1}{2}t$$

of the equation

$$\ddot{x} + \left(\frac{1}{4} - 2\epsilon b\cos^2\frac{1}{2}t\right)x + \epsilon x^3 = 0.$$

[D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers, 4th ed.*, Oxford U. Press, 2007, pp. 334.]

35. [Nov. 16.] **Stable Solution of van der Pol Equation Persists under Autonomous Perturbation.** Suppose $g \in \mathcal{C}^2(\mathbb{R}^2)$ and $\mu > 0$. Show that there is an $\varepsilon_0 > 0$ such that there is a unique periodic solution of the perturbed van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = \varepsilon g(x, \dot{x})$$

in the neighborhood of the unique solution of

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0.$$

Can you prove that this solution is orbitally asymptotically stable? [J. Hale, *Ordinary Differential Equations,* Dover 2009; repub. R. Krieger 1980; orig. J. Wiley 1969, p. 227.]

36. [Nov. 19.] **Period of the van der Pol Oscillator.** For $|\varepsilon|$ small enough, we showed that there is initial data $x(\varepsilon)$ such that the solution $\varphi(t, x(\varepsilon), \varepsilon)$ of the van der Pol equation

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0$$

is a nonconstant periodic orbit of period $T(\varepsilon)$. Find the Taylor expansion of $T(\varepsilon)$ up to second order.

[Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, p. 324.]

37. [Nov. 21.] **Resonant Forced Duffings Equation.** Prove that for $k = 9\omega^2$, the equation

$$\ddot{x} + kx + \lambda x^3 = \sigma \cos \omega t$$

has no solutions $y(t)$ of period $2\pi/\omega$ for small λ, "near" the solution $z(t) = \dfrac{\sigma}{8\omega^2} \cos \omega t$. [F. John, *Ordinary Differential Equations,* Courant Institute of Mathematical Science, 1965, p. 148.]

38. [Nov. 23.] **Autonomous Perturbation.** Let $f, g \in \mathcal{C}^1(\mathbb{R}^2)$. Consider the autonomous system

$$\begin{aligned}
\dot{x} &= \frac{x}{\sqrt{x^2 + y^2}} - x - y + \lambda f(x, y) \\
\dot{y} &= \frac{y}{\sqrt{x^2 + y^2}} + x - y + \lambda g(x, y)
\end{aligned}$$

with special periodic solution $(\cos t, \sin t)$ for $\lambda = 0$. Prove that for small $|\lambda|$, periodic solutions exist close to $(\cos t, \sin t)$. [F. John, *Ordinary Differential Equations,* Courant Institute of Mathematical Science, 1965, p. 148.]

39. [Nov. 26.] **Continuation from Harmonic Oscillator.** Show that Rayleigh's Equation has a periodic solution for small ε parameter values that is a continuations from an $\varepsilon = 0$ solution

$$\ddot{x} + \varepsilon\left(\dot{x} - \dot{x}^3\right) + x = 0.$$

[Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, pp. 318–324.]

40. [Nov. 28.] **Perihelion of Mercury.** The orbital equation of a planet about the sun is

$$\frac{d^2\, u}{d\,\vartheta^2} + u = k(1 + \varepsilon u^2)$$

where $u = \frac{1}{r}$ and r, ϑ are polar coordinates, $k > 0$ is a celestial constant and $k\varepsilon u^2$ is a relativistic correction term. Obtain a perturbation solution with initial condition $u(0) = k(e + 1)$, $\dot{u}(0) = 0$ where e is the eccentricity of the unperturbed orbit. These are the conditions at the perihelion, the nearest point to the sun on the unperturbed orbit. Show that the expansion to order ε predicts that in each orbit, the perihelion advances by $2k^2\pi\varepsilon$. [D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers, 4th ed.*, Oxford U. Press, 2007, p. 181.]

41. [Nov. 30.] **Center Manifold.** Find a center manifold for the system

$$\begin{aligned}\dot{x} &= -xy \\ \dot{y} &= -y + x^2 - 2y^2\end{aligned}$$

through the rest point at the origin. Find a differential equation for the dynamics on the center manifold. Show that every nearby solution is attracted to the center manifold. Determine the stability of the origin.

Hint: Look for a center manifold that is a graph $y = \psi(x)$ of the form

$$\psi(x) = \sum_{k=2}^{\infty} a_k \, x^k$$

using the condition of invariance $\dot{y} = \psi'(x)\dot{x}$ and $\psi(0) = \psi'(0) = 0$. Find the first few terms of the expansion, guess the rest and check. Then get the equation for the induced flow on the center manifold. [Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, p. 304.]

42. [Dec. 3.] **Bifurcation in a Genetic Control System.** Consider Griffith's model for a genetic control system, where x and y are proportional to concentration of protein and the messenger RNA from which it is translated, respectively, and $\mu > 0$ is a rate constant

$$\begin{aligned}\dot{x} &= y - \mu x \\ \dot{y} &= \frac{x^2}{1+x^2} - y.\end{aligned}$$

(a) Show that the system has three fixed points when $\mu < \mu_c$ and one when $\mu > \mu_c$where μ_c is to be determined.

(b) What is the nature of the bifurcation at $\mu = \mu_c$?

[University of Utah Ph.D. Preliminary Examination in Differential Equations, August 2012.]

43. [Dec. 5.] **Extended Center Manifold.** Find the value of μ for which there is a bifurcation at the origin for the system

$$\begin{aligned}\dot{x} &= y - x - x^2 \\ \dot{y} &= \mu x - y - y^2.\end{aligned}$$

(a) Find the evolution equation on the extended center manifold correct to third order.

(b) What is the nature of the bifurcation?

[Glendenning, *Stability, Instability and Chaos,* Cambridge U. Press, 1994, p. 246.]

44. [Dec. 7.] **Bifurcation in the Brusselator.** Show that the system undergoes a supercritical Hopf Bifurcation as the parameter passes through 2.

$$\begin{aligned}\dot{x} &= 1 - (1+\lambda)x + x^2 y. \\ \dot{y} &= \lambda x - x^2 y.\end{aligned}$$

[Y. Kuznetsov, *Elements of Applied Bifurcation Theory, 3rd. ed.*, Springer, 2004, p. 103.]