Homework for Math 6410 §1, Fall 2013

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Our main text this semester is Luis Barreira & Claudia Valls, *Ordinary Differential Equations: Qualitative Theory,* A.M.S., 2012 ISBN 978-0-8218-8749-3. Please read the relevant sections in the text as well as any cited reference. Each problem is due three class days after its assignment, or on Monday, Dec. 16, whichever comes first.

1. [Aug. 26.] **Compute a Phase Portrait using a Computer Algebra System.** This exercise asks you to figure out how to make a computer algebra system draw a phase portrait. For many of you this will already be familiar. See, *e.g.,* the MAPLE worksheet from today's lecture

$$\text{http} : //\text{www.math.utah.edu}/{\sim}\text{treiberg/M6412eg1.mws}$$
$$\text{http} : //\text{www.math.utah.edu}/{\sim}\text{treiberg/M6412eg1.pdf}$$

or my lab notes from Math 2280,

$$\text{http} : //\text{www.math.utah.edu}/{\sim}\text{treiberg/M2282L4.mws}.$$

Choose an autonomous system in the plane with at least two rest points such that one of the rest points is a saddle and another is a source or sink. Explain why your system satisfies this. (Everyone in class should have a different ODE.) Using your favorite computer algebra system, *e.g.,* MAPLE or MATLAB, plot the phase portrait indicating the background vector field and enough integral curves to show the topological character of the flow. You should include trajectories that indicate the stable and unstable directions at the saddles, trajectories at the all rest points including any that connect the nodes, as well as any seperatrices.

2. [Aug. 28.] **Stability of Fixed Points.** The SIR model of epidemics of Brauer and Castillo-Chávez relates three populations, $S(t)$ the susceptible population, $I(t)$ the infected population and $R(t)$ the recovered population. The other variables are positive constants. Assume that births in the susceptible group occur at a constant rate μK. Assume that there is a death rate of $-\mu$ for each population. Assume also that there is an infection rate of people in the susceptible population who become infected which is proportional to the contacts between the two groups βSI. There is a recovery of γI from the infected group into the recovered group. Finally, the disease is fatal to some in the infected group, which results in the removal rate $-\alpha I$ from the infected population. The resulting system of ODE's is

$$\dot{S} = \mu K - \beta SI - \mu S$$
$$\dot{I} = \beta SI - \gamma I - \mu I - \alpha I$$
$$\dot{R} = \gamma I - \mu R$$

(a) Note that the first two equations decouple and can be treated as a 2×2 system. Then the third equation can be solved knowing $I(t)$. Let $\delta = \alpha + \gamma + \mu$. For the 2×2 system, find the nullclines and the fixed points.

(b) Check the stability of the nonnegative fixed points. Show that for $\beta K < \delta$ the disease dies out. Sketch the nullclines and some trajectories in the phase plane in this case.

(c) Show that for $\beta K > \delta$ the epidemic reaches a steady state. Sketch the nullclines and some trajectories in the phase plane now.

[From R. C. Robinson, *An Introduction to Dynamical Systems*, Pearson 2004.]

3. [August 30.] **Contraction Mapping Principle and a Delay Differential Equation.**

(a) Sharpen the Contraction mapping principle: Let $E \subset \mathbf{R}^n$, $\mathcal{X} = \mathcal{C}(E, \mathbf{R}^n)$ be continuous functions and $\|u\| = \sup_{u \in E} |u(x)|$ be the sup-norm. Let $0 < b < \infty$ and $0 < k < 1$ be constants and let $T : \mathcal{X} \to \mathcal{X}$ be a transformation. Suppose that for any $\phi, \psi \in \mathcal{X}$ if $\|\psi\| \leq b$ then $\|T(\psi)\| \leq b$ and if $\|\phi\| \leq b$ and $\|\psi\| \leq b$ then

$$\|T(\psi) - T(\phi)\| \leq k\|\psi - \phi\|,$$

i.e., T is a *contraction*. Prove that there exists an element ζ with $\|\zeta\| \leq b$ such that $\zeta = T\zeta$, that is, T has a fixed point. Prove tha t ζ is the unique fixed point among points that statisfy $\|\zeta\| \leq b$. [*cf.* Perko 77[5]]

(b) The delay differential equation involves past values of the unknown function x, and so its initial data φ must be given for all times $t \leq 0$. Using the Contraction Mapping Principle (a.), show the local existence of a solution to the delay differential equation.

Theorem. Let $b > 0$. Let $f \in \mathcal{C}(\mathbb{R}^3)$ be a function that satisfies a Lipschitz condition: there is $L < \infty$ such that for all $t, x_1, x_2, y_1, y_2 \in \mathbb{R}$,

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq L(|x_1 - x_2| + |y_1 - y_2|).$$

Let $g \in \mathcal{C}(\mathbb{R})$ such that $g(t) \leq t$ for all t. Let $\varphi \in \mathcal{C}\big((-\infty, 0], \mathbb{R}\big)$ such that $|\varphi(t) - \varphi(0)| \leq b$ for all $t \leq 0$. Show that there is an $r > 0$ such that the initial value problem

$$\begin{cases} \dfrac{dx}{dt}(t) = f\big(t, x(t), x(g(t))\big) \\ \quad x(t) = \varphi(t) \qquad \text{for all } t \leq 0. \end{cases}$$

has a unique solution $x(t) \in \mathcal{C}\big((-\infty, r], \mathbb{R}\big) \cap \mathcal{C}^1\big((0, r), \mathbb{R}\big)$.

[*cf.* Saaty, Modern Nonlinear Equations, Dover 1981, §5.5.]

4. [Sept. 4.] **Escape Times.** Show that each solution $\big(x(t), y(t)\big)$ of the initial value problem

$$\begin{cases} x' = y + x^2 \\ y' = x + y^2 \end{cases} \qquad \begin{cases} x(0) = x_0 \\ y(0) = y_0 \end{cases}$$

with $x_0 > 0$ and $y_0 > 0$ cannot exist on an interval of the form $[0, \infty)$.

[*cf.* Wilson, *Ordinary Differential Equations,* Addison-Wesley, 1971, p.255.]

5. [Sept. 6.] **Generalize Gronwall's Inequality** Let $u, v, w : [a, b] \to \mathbf{R}$ be continuous functions with $w > 0$ such that

$$u(t) \leq v(t) + \int_a^t w(s)u(s)\, ds$$

for every $t \in [a, b]$. Show that

$$u(t) \le v(t) + \int_a^t w(s)v(s) \exp\left(\int_s^t w(\sigma)\, d\sigma\right) ds$$

for every $t \in [a, v]$. If in addition, v is non-decreasing, then

$$u(t) \le v(t) \exp\left(\int_a^t w(s)\, ds\right)$$

for every $t \in [a, b]$.

[*cf.* Barreira & Valls, *Ordinary Differential Equations,* A.M.S., 2012 p.54.]

6. [Sept. 9.] **Compare Solutions of Two Mathieu Equations.** One solution for Problem 19 of the 2010 Math 6410 depended on comparing the solutions of the perturbed and unperturbed problems. Find a sharp estimate for the difference in values and derivatives at $T = \frac{2\pi}{3}$ of the solutions for the two initial value problems, where u_0, u_1, ϵ are constants.

$$\ddot{x} + x = 0, \qquad\qquad\qquad \ddot{y} + (1 + \epsilon\sin(3t))y = 0,$$
$$x(0) = u_0, \qquad\qquad\qquad\qquad y(0) = u_0,$$
$$\dot{x}(0) = u_1; \qquad\qquad\qquad\qquad \dot{y}(0) = u_1.$$

Let $y(t; u_0, u_1, \epsilon)$ solve the IVP. Use your estimate to show $|y(T; 1, 0, \epsilon) + \dot{y}(T; 0, 1, \epsilon)| < 2$ for small $|\epsilon|$.

7. [Sept. 11.] **Nagumo's Uniqueness Theorem.** Prove the theorem and show that it implies the uniqueness statement in the Picard Theorem.

Theorem. [Nagumo, 1926] Suppose $f \in C(\mathbb{R}^2)$. Then the initial value problem

$$\frac{dy}{dt} = f(t, y),$$
$$y(0) = 0,$$

has a solution by Peano's Existence Theorem. Assume in addition that

$$|f(t, y) - f(t, z)| \le \frac{|y - z|}{|t|}$$

for all $t, y, z \in \mathbb{R}$ such that $t \ne 0$. Then the solution is unique.

8. [Sept. 13.] **Comparison Principle.** Suppose that $J = (a, b)$ is an interval, $t_0 \in J$ and $f(t, x) \in \mathcal{C}(\mathbf{R}^2)$ is a function which is locally Lipschitz with respect to x in $\mathbf{R}^2$. Let $x \in \mathcal{C}^1(J)$ be a solution of

$$\frac{dx}{dt}(t) = f\big(t, x(t)\big).$$

Suppose $y : J \to \mathbf{R}$ is a differentiable function such that

$$\frac{dy}{dt}(t) \ge f\big(t, y(t)\big), \qquad \text{for all } t \in J,$$
$$y(t_0) \ge x(t_0).$$

Prove that $y(t) \ge x(t)$ for all $t_0 \le t < b$.

9. [Sept. 16.] **Eternal Solutions.** Let $f \in \mathcal{C}^1(\mathbf{R}^n, \mathbf{R}^n)$ and let $\langle \bullet, \bullet \rangle$ be an inner product of $\mathbf{R}^n$. Consider the initial value problem

$$\frac{dx}{dt} = f(x)$$

$$x(t_0) = x_0$$

(1)

(a) Show that if $\langle f(x), x \rangle \leq 0$ for every $x \in \mathbf{R}^n$ then each solution of (1) is defined for all $t \geq t_0$.

(b) Suppose that $g \in \mathcal{C}^1(\mathbf{R}^n)$ such that

$$g(x) \geq \langle x, x \rangle \qquad \text{and} \qquad \langle f(x), \nabla g(x) \rangle < 0$$

for all $x \in \mathbf{R}^n \backslash \{0\}$, then each solution of (1) is defined for all $t \geq t_0$.

[*cf.* Barreira & Valls, *Ordinary Differential Equations*, A.M.S., 2012 p.54.]

10. [Sept. 18.] **Concrete Variational Equation.** Let

$$f \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 + x_1^2 \\ x_3 + x_1^2 \end{pmatrix}.$$

Find the solution $\varphi(t, y) \in \mathbf{R}^3$ of

$$\frac{dx}{dt} = f(x(t)),$$

$$x(0) = y.$$

Find

$$\Phi(t, y) = D_2\, \varphi(t, y).$$

Show that it satisfies the variational equation

$$\frac{d\Phi}{dt} = Df(\varphi(t, y)) \cdot \Phi(t, y),$$

$$\Phi(0) = I.$$

[Perko, p. 84.]

11. [Sept. 20.] **Find a Periodic Solution.** This exercise gives conditions for an ordinary differential equation to admit periodic solutions.

(a) Let $J = [0, 1]$ denote an interval and let $\phi \in C(J, J)$ be a continuous transformation. Show that ϕ admits at least one fixed point. (A fixed point is $y \in J$ so that $\phi(y) = y$.)

(b) Assume that $f \in C(\mathbb{R} \times [-1, 1])$ such that for some $L < \infty$ and some $0 < T < \infty$ we have

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|,$$

$$f(T + t, y_1) = f(t, y_1),$$

$$f(t, -1)f(t, +1) < 0$$

for all $t \in \mathbf{R}$ and all $y_1, y_2 \in [-1, 1]$. Using (a), show that the equation $y' = f(t, y)$ has at least one solution periodic of period T.

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(c) Apply (b) to $y' = a(t)y + b(t)$ where a, b are T periodic functions.

12. [Sept. 23.] **Existence via Schauder's Theorem.** Give another proof of the Peano Existence Theorem using the Schauder Fixed Point Theorem.

Theorem. [Peano Existence Theorem] Let $D \subset \mathbf{R} \times \mathbf{R}^n$ be a domain and $f \in \mathcal{C}(D, \mathbf{R}^n)$. Then for any $(t_0, x_0) \in D$ there is $\epsilon > 0$ and a continuously differentiable function $x(t) : [t_0 - \epsilon, t_0 + \epsilon] \to \mathbf{R}^n$ which solves the initial value problem

$$\frac{dx}{dt} = f(t, (x(t))), \qquad \text{for all } t \in [t_0 - \epsilon, t_0 + \epsilon];$$
$$x(t_0) = x_0.$$

Theorem. [Schauder Fixed Point Theorem] Let $\mathcal{A}$ be a closed, bounded, convex subset of a Banach space $\mathcal{X}$ and $T : \mathcal{A} \to \mathcal{A}$ be a completely continuous function. Then T has a fixed point in $\mathcal{A}$.

A subset $\mathcal{K}$ of a Banach space is *compact* if any sequence in $\{\phi_i\}_{i=1,2,\dots} \subset \mathcal{K}$ has a subsequence that converges to an element in $\mathcal{K}$. f is *compact* if for any bounded set $\mathcal{B} \subset \mathcal{X}$, the closure of the set $f(\mathcal{B})$ is compact. f is *completely continuous* if it is both compact and continuous. [*cf.* Hale, p. 14.]

13. [Sept. 25] **Application of Liouville's Theorem.** Find a solution of the IVP for Bessel's Equation of order zero

$$\begin{cases} x'' + \dfrac{1}{t}\, x' + x = 0 \\ x(0) = 1, \qquad x'(0) = 0 \end{cases}$$

by assuming the solution has a power series representation (or use Frobenius Method.) Use Liouville's formula for the Wronskian to find a differential equation for a second linearly independent solution of the differential equation. Show that this solution blows up like $\log t$ as $t \to 0$. [*cf.* Fritz John, Ordinary Differential Equations, Courant Institute of Mathematical Sciences, 1965, p. 90.]

14. [Sept. 27.] **Real Canonical Form.** Let A be a real 2×2 matrix whose eigenvalues are $a \pm bi$ where $a, b \in \mathbb{R}$ such that $b \neq 0$. Show that there is a real matrix Q so that

$$Q^{-1}AQ = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Use this fact to solve the system

$$\begin{aligned} x' &= -13x - 10y \\ y' &= 20x + 15y \end{aligned}$$

15. [Oct. 4.] **Jordan Form.** Find the generalized eigenvectors, the Jordan form and the general solution

$$\dot{\mathbf{y}} = \begin{pmatrix} 6 & 6 & 4 \\ -2 & -2 & -4 \\ 2 & 6 & 8 \end{pmatrix} \mathbf{y}.$$

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16. [Oct.2.] **Just Multiply by** t. Consider the n-th order constant coefficient linear homogeneous scalar equation

$$x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_1 x' + a_0 x = 0$$

where a_i are complex constants. Convert to a first order differential system $\mathbf{x}' = A\mathbf{x}$. Show that the geometric multiplicity of every eigenvalue of A is one. Show that a basis of solutions is $\{t^k \exp(\mu_i t)\}$ where $i = 1, \ldots, s$ correspond to the distinct eigenvalues μ_i and $0 \le k < m_i$ where m_i is the algebraic multiplicity of μ_i. [*cf.*, Gerald Teschl, *Ordinary Differential Equations and Dynamical Systems*, Amer. Math. Soc., 2012, p.68.]

17. [Oct. 4.] **To Use Jordan Form or Not to Use Jordan Form.** Sometimes the use of the Jordan Canonical Form and matrices with multiple eigenvalues can be avoided using the following considerations.

(a) Let $A \in \mathcal{M}_{n \times n}(\mathbb{C})$. Show that given $\epsilon > 0$ there exists a matrix B with distinct eigenvalues so that $\|A - B\| \le \epsilon$.

(b) Give three proofs of $\det(e^A) = e^{\operatorname{trace}(A)}$.

(c) Let $A \in \mathcal{M}_{n \times n}(\mathbb{C})$. By a simpler algorithm than finding the Jordan Form, one can change basis by a P that transforms A to upper triangular

$$P^{-1}AP = U = \begin{pmatrix} u_{11} & u_{12} & \ldots & u_{1n} \\ 0 & u_{22} & \ldots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & u_{nn} \end{pmatrix}. \tag{2}$$

Show that this fact can be used instead of Jordan Form to characterize all solutions of $\dot{y} = Ay$ (as linear combinations of products of certain exponentials, polynomials and trigonometric functions). [*c.f.*, Bellman, *Stability Theory of Differential Equations*, pp. 21–25.])

(d) Let $A \in \mathcal{M}_{n \times n}(\mathbb{C})$. Show that given $\epsilon > 0$ there exists a nonsingular P such that in addition to (2) we may arrange that $\sum_{i<j} |u_{ij}| < \epsilon$.

(e) [Optional.] Find all continuous scalar valued functions $f \in C(\mathcal{M}_{n \times n}(\mathbb{C}), \mathbb{C})$ so that

$$f(AB) = f(A)f(B) \qquad \text{for all } A, B.$$

You can probably find several different arguments on your own. [*ibid.*; or Kurosh, *Higher Algebra*, p. 334.]

18. [Oct. 7.] **Variation of Parameters Formula.** Solve the inhomogeneous linear system

$$\begin{cases} \dot{\mathbf{x}} = A(t)\,\mathbf{x} + \mathbf{b}(t), \\ \mathbf{x}(t_0) = \mathbf{c}; \end{cases}$$

where

$$A(t) = \begin{pmatrix} -2\cos^2 t & -1 - \sin 2t \\ 1 - \sin 2t & -2\sin^2 t \end{pmatrix}, \qquad \mathbf{b}(t) = \begin{pmatrix} 1 \\ e^{-2t} \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Hint: a fundamental matrix is given by

$$U(t,0) = \begin{pmatrix} e^{-2t}\cos t & -\sin t \\ e^{-2t}\sin t & \cos t \end{pmatrix}.$$

[*cf.* Perko, Differential Equations and Dynamical Systems, Springer, 1991, p. 62.]

19. [Oct. 9.] **Bounded Solutions of Hill's Equation.** Show that if $|\varepsilon|$ is sufficiently small, then all solutions are bounded

$$\ddot{u} + \big[1 + \varepsilon \sin(3t)\big]u = 0.$$

[U. Utah Preliminary Examination in Differential Equations, January 2004.]

20. [Oct. 11.] **Periodic Inhomogeneous Equations.** Suppose that the $A(t)$ and $b(t)$ are continuous $T > 0$ periodic matrix and vector functions such that none of the characteristic multipliers of

$$y' = A(t)y$$

equals one. Then there is a unique T-periodic solution (which may be identically zero) to the inhomogeneous equation

$$y' = A(t)y + b(t).$$

[*cf.* Cronin, Ordinary Differential Equations, Introduction to Qualitative Theory, 3rd ed., Chapman & Hall/CRC, 2008, p. 84.]

21. [Oct. 21.] **Polar Coordinates.** Let a, b and p be positive constants. Consider the differential equation

$$\dot{x} = -\frac{ax}{\sqrt{x^2 + y^2}}$$
$$\dot{y} = -\frac{ay}{\sqrt{x^2 + y^2}} + b$$

which models the flight of a bird heading toward the origin at constant speed a, that is moved off course by a steady wind of velocity b. Determine the conditions on a and b to ensure that the solution starting at $(p, 0)$ reaches the origin. Hint: change to polar coordinates and study the phase portrait of the differential equation on the cylinder. [Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, p. 86.]

22. [Oct. 23.] **Condition for Asymptotic Stability.** Suppose that the zero solution to $\dot{x} = Ax$ is asymptotically stable. Let $g(t, x) \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ satisfy $g(t, 0) = 0$ and

$$|g(t, x)| \leq h(t)|x|, \qquad \text{for all } t \geq 0 \text{ and } x \in \mathbb{R}^n,$$

where $h(t)$ satisfies for positive constants k and r,

$$\int_0^t h(s)\,ds \leq kt + r, \qquad \text{for all } t \geq 0.$$

Show that there is a constant $k_0(A) > 0$ such that if $k \leq k_0$, then the zero solution of

$$\dot{x} = Ax + g(t, x)$$

is asymptotically stable. [*cf.* James H. Liu, *A First Course in the Qualitative Theory of Differential Equations,* Prentice Hall 2003, p. 243.]

23. [Oct. 25.] **Liapunov Functions.** Use a Liapunov Function to show that the zero solution is asymptotically stable

$$\ddot{x} + (2 + 3x^2)\,\dot{x} + x = 0.$$

Hint: Show that this equation is equivalent to the system

$$\dot{x} = y - x^3$$
$$\dot{y} = -x + 2x^3 - 2y.$$

[D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers, 4th ed.*, Oxford U. Press, 2007, pp. 348–349.]

24. [Oct. 28.] **Četaev's Theorem.** Show that the zero solution is not stable.

$$\dot{x} = x^3 + xy$$
$$\dot{y} = -y + y^2 + xy - x^3.$$

[J. Hale and H. Koçek, *Dynamics and Bifurcations*, Springer 1991, p. 286.]

25. [Oct. 30.] **La Salle's Invariance Principle.** Assume that the continuously differentiable functions satisfy $f(x) > 0$ and $xg(x) > 0$ for $x \neq 0$. Use La Salle's Invariance Principle to show that the zero solution of

$$\ddot{x} + f(x)\,\dot{x} + g(x) = 0$$

is asymptotically stable. [Brauer & Nohel, *The Qualitative Theory of Ordinary Differential Equations: An Introduction.*, Dover, 1989.]

26. [Nov. 1.] **Asymptotically Stable Equilibrium in a Discrete Dynamical System.**

   (a) Let A be a real $n \times n$ matrix such that $|\lambda| < \gamma$ for all eigenvalues λ of A. Show that there is a norm $\| \cdot \|$ on $\mathbb{R}^n$ so that $\|Ax\| \leq \gamma \|x\|$ for all $x \in \mathbb{R}^n$.

   (b) Let $P \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ such that $P(0) = 0$ and $|\lambda| < 1$ for all eigenvalues of $DP(0)$. Show that 0 is an asymptotically stable fixed point of the discrete dynamical system in $\mathbb{R}^n$

$$x_{n+1} = P(x_n),$$
$$x_1 = \xi.$$

27. [Nov. 4.] **Topological Conjugacy.** Let

$$A = \begin{pmatrix} -\alpha & \beta \\ -\beta & -\alpha \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \alpha > 0.$$

Explicitly construct the conjugacy found in the proof of Theorem 2.50. [*cf.*, Gerald Teschl, *Ordinary Differential Equations and Dynamical Systems,* Amer. Math. Soc., 2012, p.270.]

28. [Nov. 6.] **Hartman-Grobman Theorem.** Find a homeomorphism h in a neighborhood of 0 that establishes an topological conjugacy between the flow of the differential system and the flow of the linearized system, *i.e.*, $h(\psi(t, x)) = e^{tA}h(x)$ where $A = Df(0)$ and $\psi(t, x_0)$ is the solution of $\dot{\mathbf{x}} = f(\mathbf{x})$, the nonlinear system given by

$$\dot{x} = -x$$
$$\dot{y} = -y + xz,$$
$$\dot{z} = z.$$

[In §4.2, Barreira & Valls discuss a proof, but you can guess h from the solutions and verify.]

29. [Nov 8.] **International Whaling Commission Model.** A simple rescaled difference equation for modeling the population u_n of sexually mature baleen whales is

$$u_{n+1} = su_n + R(u_{n-T}),$$

where $0 < s < 1$ is the survival parameter, T is an integer corresponding to time to sexual maturity, and $R(u_{n-T})$ is the recruitment function that augments the adult population from births T years earlier. If

$$R(u) = (1-s)[1 + q(1-u)]u,$$

where $q > 0$ describes the fecundity increase due to low density and the delay is $T = 1$, derive the condition for a positive steady state u^* to be stable and find for which q it holds. [J. D. Murray, *Mathematicsl Biology,* Biomathematics Texts 19, Springer 1989, p. 62.]

30. [Nov. 11.] **Small Perturbation of an Invertible Linear Map.** Syppose A is an invertible real $n \times n$ matrix. Suppose that $\alpha : \mathbf{R}^n \to \mathbf{R}^n$ satisfies

$$|\alpha(x) - \alpha(y)| \le \delta|x - y|$$

for all $x, y \in \mathbf{R}^n$. Show that there is $\varepsilon > 0$ so that if $\delta < \varepsilon$ then the map

$$h(x) = Ax + \alpha(x) : \mathbf{R}^n \to \mathbf{R}^n$$

is a homeomorphism.

[Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, p. 315.]

31. [Nov. 13.] **Stable and Unstable Manifolds.** Find the stable manifold W^s and unstable manifold W^u near the origin of the system

$$\dot{x} = -x$$
$$\dot{y} = -y + x^2$$
$$\dot{z} = z + y^2.$$

[*cf.* Perko, Differential Equations and Dynamical Systems, Springer, 1991, p. 116–117.]

32. [Nov. 15.] **Traveling Wave.** For constant $r > 0$ let $u(x, t)$ be a real valued function satisfying Fisher's Equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + ru(1 - u), \qquad \text{for } x \in \mathbf{R} \text{ and } t \ge 0.$$

which models the spread of disease. A special solution is the *traveling wave,* $u(t, x) = v(x - ct)$ where $c > 0$ is the wave speed.

(a) Show that the traveling wave satisfies the ODE

$$v'' + cv' + rv(1 - v) = 0,$$

where $v' = dv/ds$ and $s = x - ct$.

(b) Show that for every $c \ge 2\sqrt{r}$, Fischer's Equation has a traveling wave solution satisfying $v(s) \to 1$ as $s \to -\infty$ and $v(s) \to 0$ as $s \to \infty$ with $v'(s) < 0$ for all s. [Hint: Discuss the stability properties of equilibrium points. Use special properties of the unstable manifold at $(1, 0)$. Show there is a forward invariant triangle in the (v, v') plane bounded by the lines $v' = 0$, $v = 1$ and $v' = -\mu v$ for appropriate $\mu > 0$.]

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[*cf.* J. Hale and H. Koçek, *Dynamics and Bifurcations*, Springer 1991, p. 299.]

33. [Nov. 18.] **Brusselator System.** Show that there is a non-constant periodic trajectory for the system

$$\dot{x} = 1 - 4x + x^2 y$$
$$\dot{y} = 3x - x^2 y$$

[University of Utah Ph.D. Preliminary Examination in Differential Equations, August 2004.]

34. [Nov. 20.] **Periodic Orbits in Three Space.** Verify that there exists an autonomous equation in $\mathbf{R}^3$ with a periodic orbit but without fixed points.

[*cf.* Barreira & Valls, *Ordinary Differential Equations,* A.M.S., 2012 p. 197.]

35. [Nov. 22.] **Dulac's Criterion.** Prove the following theorem.
Theorem. Let $A \subset \mathbb{R}^2$ be an annular domain. Let $f \in \mathcal{C}^1(A, \mathbb{R}^2)$ and let $\rho \in \mathcal{C}^1(A, \mathbb{R})$. Show that if $\mathrm{div}(\rho f)$ is not identically zero and does not change signs in any open subset of A then the equation $x' = f(x)$ has at most one periodic solution in A.

Use this to show that the van der Pol oscillator $(\lambda = \mathrm{const.} \neq 0)$

$$\dot{x} = y$$
$$\dot{y} = -x + \lambda(1 - x^2)y$$

has at most one limit cycle in the plane. Hint: let $\rho = (x^2 + y^2 - 1)^{-1/2}$. [Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, p. 90.]

36. [Nov. 25.] **Periodic Orbit in Predator-Prey System.** A generalization of a predator-prey system given by Brauer and Castillo-Chavez is

$$\dot{x} = x\left(1 - \frac{x}{30} - \frac{y}{x + 10}\right),$$
$$\dot{y} = y\left(\frac{x}{x + 10} - \frac{1}{3}\right).$$

(a) Show that the fixed points are $(0,0)$, $(30,0)$ and $(5, 12.5)$ and have saddle, saddle, source type, resp.

(b) Show that the region bounded by $0 \le x$, $0 \le y$ and $x + y \le 50$ is forward invariant.

(c) Show that there is no orbit Γ with $\alpha(\Gamma) = \{(30,0)\}$ and $\omega(\Gamma) = \{(0,0)\}$. Conclude that there is a nonconstant periodic orbit.

[R. C. Robinson, *An Introduction to Dynamical Systems Continuous and Discrete*, Pearson/Prentice Hall, 2004, p. 238.]

37. [Nov. 9.] **Stability of a Periodic Orbit.** A periodic solution to the system

$$\dot{x} = x - 4y - \frac{1}{4}x^3 - xy^2$$
$$\dot{y} = x + y - \frac{1}{4}x^2 y - y^3$$
$$\dot{z} = -z,$$

is $\gamma(t) = (2\cos 2t, \sin 2t, 0)$. Is $\gamma(t)$ orbitally asymptotically stable? Choose a transverse planar section L. Determine the Poincaré map $\wp : L \to L$ and find its differential at the periodic orbit. In particular, find the variational equation, the fundamental matrix associated with the periodic orbit and compute the Floquet Multipliers. [*cf.* Perko, Differential Equations and Dynamical Systems, Springer, 1991, p. 213.]

38. [Dec. 2.] **Stability of a Periodic Orbit.** Let $f \in \mathcal{C}^2(\mathbf{R}^2)$ be such that zero is a *regular value* ($\nabla f(x) \neq 0$ whenever $f(x) = 0$.) Then the zero level set

$$A = \left\{ (x, y) \in \mathbf{R}^2 : f(x, y) = 0 \right\}$$

consists of embedded $\mathcal{C}^2$ curves. Prove that each bounded component of A is an attracting hyperbolic limit cycle for the differential equation

$$\dot{x} = -f_y - f\, f_x,$$
$$\dot{y} = f_x - f\, f_y.$$

Describe the phase portrait in case

$$f(x, y) = \left((x - \mu)^2 + y^2 - 1 \right) \left(x^2 + y^2 - 9 \right).$$

[Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, p. 197.]

39. [Dec. 4.] **Perturbation of Stable Solution of van der Pol Equation.** Suppose $g \in \mathcal{C}^2(\mathbb{R}^2)$ and $\mu > 0$. Show that there is an $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$ there is a unique periodic solution of the perturbed van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = \varepsilon g(x, \dot{x})$$

in the neighborhood of the unique solution of

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0.$$

Can you prove that this solution is orbitally asymptotically stable? [J. Hale, *Ordinary Differential Equations,* Dover 2009; repub. R. Krieger 1980; orig. J. Wiley 1969, p. 227.]

40. [Dec. 6.] **Center Manifold.** Find a center manifold for the system

$$\dot{x} = -xy$$
$$\dot{y} = -y + x^2 - 2y^2$$

through the rest point at the origin. Find a differential equation for the dynamics on the center manifold. Show that every nearby solution is attracted to the center manifold. Determine the stability of the origin.

Hint: Look for a center manifold that is a graph $y = \psi(x)$ of the form

$$\psi(x) = \sum_{k=2}^{\infty} a_k \, x^k$$

using the condition of invariance $\dot{y} = \psi'(x)\dot{x}$ and $\psi(0) = \psi'(0) = 0$. Find the first few terms of the expansion, guess the rest and check. Then get the equation for the induced flow on the center manifold. [Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, p. 304.]

41. [Dec. 9.] **Bifurcation in a Genetic Control System.** Consider Griffith's model for a genetic control system, where x and y are proportional to concentration of protein and the messenger RNA from which it is translated, respectively, and $\mu > 0$ is a rate constant

$$\dot{x} = y - \mu x$$
$$\dot{y} = \frac{x^2}{1 + x^2} - y.$$

(a) Show that the system has three fixed points when $\mu < \mu_c$ and one when $\mu > \mu_c$ where μ_c is to be determined.

(b) What is the nature of the bifurcation at $\mu = \mu_c$?

[University of Utah Ph.D. Preliminary Examination in Differential Equations, August 2012.]

42. [Dec. 11.] **Extended Center Manifold.** Find the value of μ for which there is a bifurcation at the origin for the system

$$\dot{x} = y - x - x^2$$
$$\dot{y} = \mu x - y - y^2.$$

(a) Find the evolution equation on the extended center manifold correct to third order.

(b) What is the nature of the bifurcation?

[Glendenning, *Stability, Instability and Chaos,* Cambridge U. Press, 1994, p. 246.]

43. [Dec. 13.] **Bifurcation in the Brusselator.** Show that the system undergoes a super-critical Hopf Bifurcation as the parameter passes through 2.

$$\dot{x} = 1 - (1 + \lambda)x + x^2 y.$$
$$\dot{y} = \lambda x - x^2 y.$$

[Y. Kuznetsov, *Elements of Applied Bifurcation Theory, 3rd. ed.,* Springer, 2004, p. 103.]