Homework for Math 6510 §1, Fall 2018

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Our main text this semester is Kevin Wortman's, *Math 6510 Notes,*
`http://www.math.utah.edu/ wortman/6510.pdf`.
Please read the relevant sections in the text as well as any cited reference. Each problem is due three class days after its assignment, or on Friday, Dec. 7, whichever comes first. Please write the question on your solutions.

1. [Aug. 20] **Real projective space.** Show that $\mathbb{RP}^n$ is Hausdorff, 2nd countable and compact. [John M. Lee, *Introduction to Smooth Manifolds, 2nd ed.*, Springer 2013, p. 7.]

2. [Aug. 22] **σ-compactness.** A topological space is called *σ-compact* if it can be expressed as a countable union of compact subspaces. Show that a locally Euclidean Hausdorff space is a topological manifold if and only if it is σ-compact. [John M. Lee, *Introduction to Smooth Manifolds, 2nd ed.*, Springer 2013, p. 30.]

3. [Aug. 24] **Differential structures of a graph.**

   (a) Let M and N be two topological manifolds and $f : M \to N$ continuous. The graph is defined to be
   $$\Gamma = \{(x, f(x)) : x \in M\} \subset M \times N.$$
   If M is a $\mathcal{C}^r$-manifold and (U, φ) a coordinate chart from a $\mathcal{C}^r$ atlas, let
   $$W = \{(x, f(x)) : x \in U\}$$
   and $\zeta = \varphi \circ \pi_1$, where $\pi_1(x, y) = x$ is the projection onto the first factor. Show that $\{(W, \zeta)\}$ is a $\mathcal{C}^r$ atlas for Γ.

   (b) Now suppose $f : M \to M$ is a homeomorphism. Then $\Gamma \subset M \times M$ is both the graph of f and the graph of f^{-1}
   $$\Gamma = \{(x, f(x)) : x \in M\} = \{(f^{-1}(y), y) : y \in M\}.$$
   This gives two $\mathcal{C}^r$ structures for Γ as in (a). Find necessary and sufficient conditions that the two atlases have the same maximal extensions. [Siavash Shahshahani, *An Introductory Course on Differentiable Manifolds,* Dover 2016, p. 122.]

4. [Aug. 27] **Smooth functions.** Assume $a < b$.

   (a) Show $f \in \mathcal{C}^\infty(\mathbb{R})$ where
   $$f(x) = \begin{cases} \exp\left(-\frac{1}{x^2}\right), & \text{if } x > 0; \\ 0, & \text{if } x \le 0. \end{cases}$$

(b) Show $g(x) = f(x-a)f(b-x)$ is a smooth function which is positive on (a,b) and zero elsewhere. Put

$$h(x) = \frac{\int_{-\infty}^{x} g(x)\,dx}{\int_{-\infty}^{\infty} g(x)\,dx}.$$

Show $h(x)$ is a smooth function such that $h(x) = 0$ for $x \le a$, $0 < h(x) < 1$ for $a < x < b$ and $h(x) = 1$ for $x \ge b$.

(c) Construct a smooth function on $\mathbb{R}^n$ that equals one on a closed ball $\overline{B_a(x)}$, is zero off $B_b(x)$ and is strictly between zero and one on intermediate points, where $0 < a < b$ and $x \in \mathbb{R}^n$.

(d) Let $a_0, a_1, a_2, \ldots$ be an arbitrary sequence of real numbers. Prove that there exists a function $f \in \mathcal{C}^\infty(\mathbb{R})$ such that the n-th derivative $f^{(n)}(0) = a_n$ for all $n = 0, 1, 2, \ldots$.

(e) Let $U \subset \mathbb{R}^n$ be an open set and $C \subset U$ be closed. Show that there is $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that $0 \le f(x) \le 1$, $C \subset f^{-1}(1)$ and $\operatorname{spt} f \subset U$ where the *support* is

$$\operatorname{spt}(f) = \overline{\{x \in \mathbb{R}^n : f(x) \ne 0\}}.$$

(f) Show that in fact, the function from (e) can be chosen so that $f^{-1}(1) = C$.

5. [Aug. 29.] **Space of matrices of fixed rank.** Prove that the set of $m \times n$ real matrices of rank r is an analytic manifold of dimension $r(m+n-r)$. Hint: after multiplication by permutation matrices P and Q, such matrix has the form

$$PMQ = \begin{array}{cc} & \begin{array}{cc} r & n-r \end{array} \\ \begin{array}{c} r \\ m-r \end{array} & \begin{pmatrix} B & C \\ D & E \end{pmatrix} \end{array}$$

where the $r \times r$ matrix B is nonsingular. Then prove that $\operatorname{rank} M = r$ if and only if $E - DB^{-1}C = 0$. [Guillemin & Pollack, *Differential Topology,* Prentice Hall, 1974, p. 27.]

6. [Aug. 31.] **Smooth structure on a closed surface.** Show that every closed surface can be given a smooth structure. [Hint: A closed surface is homeomorphic to either one of the oriented surface Σ_g's of genus $g = 0, 1, 2, \ldots$, the sphere $\Sigma_0 = \mathbb{S}^2$ or the identification spaces Σ_g of a $4g$-gon whose sides are glued according to the word $a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$ or one of the nonoriented surfaces of genus $g = 1, 2, 3, \ldots$ the identification spaces N_g of a $2g$-gon whose sides are glued according to the word $a_1 a_1 a_2 a_2 \cdots a_g a_g$ (Figure 1). Warning, see Spivak, *Differential Geometry*, vol. 1, Publish or Perish, 1970, p. 2-39, prob. 14.]

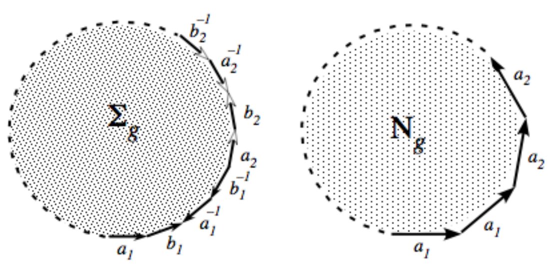

Figure 1: Surfaces as identification spaces for Problem 6.

7. [Sept. 5.] **Smooth Maps.** For each of the following maps between spheres, compute sufficiently many coordinate representations to prove that it is smooth.

   (a) Power map: $p_n : \mathbb{S}^1 \to \mathbb{S}^1$ given by $p_n(z) = z^n$ where $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ and $n \in \mathbb{Z}$.

   (b) Antipodal map: $a : \mathbb{S}^n \to \mathbb{S}^n$ given by $a(x) = -x$.

   (c) $F : \mathbb{S}^3 \to \mathbb{S}^2$ given by $F(w,z) = (z\bar{w} + w\bar{z},\, iw\bar{z} - iz\bar{w},\, z\bar{z} - w\bar{w})$ where $\mathbb{S}^3 = \{(w,z) \in \mathbb{C}^2 : |w|^2 + |z|^2 = 1\}$.

   [John M. Lee, *Introduction to Smooth Manifolds, 2nd ed.*, Springer, 2013, p. 48.]

8. [Sept. 7.] **Tangent Vectors.** Let (x,y) denote standaard coordinates of $\mathbb{R}^2$. Verify that $(\tilde{x}, \tilde{y})$ are global smooth coordinate in $\mathbb{R}^2$ where

$$\tilde{x} = x, \qquad \tilde{y} = y + x^3.$$

   Let p be the point $(1,0) \in \mathbb{R}^2$ in standard coordinates, and show that

$$\left.\frac{\partial}{\partial x}\right|_p \neq \left.\frac{\partial}{\partial \tilde{x}}\right|_{\tilde{p}},$$

   even though the coordinate functions x and $\tilde{x}$ are identically equal. [John M. Lee, *Introduction to Smooth Manifolds, 2nd ed.*, Springer, 2013, p. 65.]

9. [Sept, 10.] **Tangent Vector in Stereographic Coordinates.** Consider stereographic coordinate charts (see p. 23) for the sphere $\{(\mathbb{S}^2 - N, \varphi_1), (\mathbb{S}^2 - S, \varphi_2)\}$ where $N = (0,0,1)$, $S = (0,0,-1)$ and

$$(y^1, y^2) = \varphi_1(x^1, x^2, x^3) = \frac{(x^1, x^2)}{1 - x^3}, \qquad (z^1, z^2) = \varphi_2(x^1, x^2, x^3) = \frac{(x^1, x^2)}{1 + x^3}.$$

   If $p \in \mathbb{S}^2 - \{N, S\}$ and the vector at p in the φ_1 chart is $v^i \dfrac{\partial}{\partial y^i}$, what is it in the φ_2 chart? [Wortman, *Math 6510 Notes,* p. 34.]

10. [Sept. 12.] **Submersion of a Compact Manifold.** Let M be a nonempty smooth compact manifold. Show that there is no smooth submersion $f : M \to \mathbb{R}^k$ for any $k > 0$. [John M. Lee, *Introduction to Smooth Manifolds, 2nd ed.*, Springer, 2013, p. 96.]

11. [Sept. 14.] **Proper Immersion.** A continuous map between topological spaces is called *proper* if the inverse image of any compact set is compact. Show that if $f : M \to N$ is a continuous, proper and injective then $f : M \to f(M)$ is a homeomorphism. Let $g : M \to N$ be a smooth injective proper immersion. Then g is an embedding. [Wortman, *Math 6510 Notes,* p. 47.]

12. [Sept. 17.] **Stiefel Manifold.** A k-frame is an ordered set of k independent vectors in $\mathbb{R}^n$. The set of orthonormal k-frames, denoted $V_{n,k}$, is called the *Stiefel Manifold.* Let $L(n,k)$ denote the real $n \times k$ matrices, $L(n) = L(n.n)$ and $S(n)$ the symmetric matrices in $L(n)$. Let $\alpha_{n,k}(A) := A^T A$ where A^T means transpose.

   (a) Prove $\alpha_{n.k}(A) = I_k$ (identity matrix) if and only if the columns of A are orthonormal in $\mathbb{R}^n$. Then $V_{n,k} = \alpha_{n,k}^{-1}(I_k)$. Hence $V_{n,1} = \mathbb{S}^{n-1} \subset \mathbb{R}^n$ is the sphere and $V_{n,n} = O(n)$ is the orthogonal group.

   (b) Compute the differential $d(\alpha_{n,k})_A$ at an arbitrary $A \in L(n,k)$. Show that I_k is a regular value of $\alpha_{n,k}$ so $V_{n,k}$ is a manifold. What is the tangent space of $V_{n,k}$ at each of its points? Prove that $V_{n,k}$ is compact.

13. [Sept. 19.] **Fundamental Theorem of Algebra.** Let

$$p(z) = z^m + a_1 z^{m-1} + \cdots + a_m$$

be a polynomial with complex coefficients.

(a) Show that the map $p : \mathbb{C} \to \mathbb{C}$ is a local diffeomorphism except at finitely many points. [Guillemin & Pollack, *Differential Topology,* Prentice Hall, 1974, p. 26.]

(b) Let $\sigma : \mathbb{S}^2 - \{(0,0,1)\} \to \mathbb{R}^2 \cong \mathbb{C}$ be stereographic projection. Let $f(x) = \sigma^{-1} \circ p \circ \sigma(x)$ if $x \neq (0,0,1)$ and $f(0,0,1) = (0,0,1)$. Show that $f : \mathbb{S}^2 \to \mathbb{S}^2$ is smooth.

(c) Show that f has finitely many critical points on $\mathbb{S}^2$.

(d) Using the fact that the number of preimage points $\sharp f^{-1}(y)$ is locally constant and the fact that the set of regular values is connected in $\mathbb{S}^2$, conclude the Fundamental Theorem of Algebra: every nonconstant polynomial $p(z)$ must have a zero. [Milnor, *Topology from the Differential Viewpoint,* U. Va. Press, 1965, p. 9.]

14. [Sept. 21.] **Normal Form for Immersions.** Suppose $f : M^m \to N^n$ is a smooth immersion. Show that for all $p \in M$ there are local coordinates x for M at p and y for N at $f(p)$ so that in these coordinates f is given by

$$y(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0) \in \mathbb{R}^n.$$

[Wortman, *Math 6510 Notes,* p. 58.]

15. [Sept. 24.] **Transverse maps.** Let $f(x,y) = (e^y \cos x, e^y \sin x, e^{-y})$ and $\mathbb{S}^2(r)$ be the standard two sphere of radius r embedded in $\mathbb{R}^3$.

(a) For which r is f trasverse to $\mathbb{S}^2(r)$?

(b) For which r is $f^{-1}(\mathbb{S}^2(r))$ an embedded submanifold of $\mathbb{R}^3$?

[John M. Lee, *Introduction to Smooth Manifolds, 2nd ed.*, Springer, 2013, p. 148.]

16. [Sept. 26.] **Tangent Space to the Pulled Back Manifold.** Let $f : M^m \to N^n$ be a smooth map of smooth manifolds. Let $Z^z \subset M^m$ and $Q^q \subset N^n$ be smooth embedded submanifolds. Show:

(a) If $f(Z) \subset Q$ and $p \in Z$ then $df_p(T_p Z) \subset T_{f(p)} Q$.

(b) If $y \in N$ is a regular value for f and $p \in f^{-1}(y)$ then $T_p(f^{-1}(y)) = (df_p)^{-1}(0)$.

(c) If $f \pitchfork Q$ and $p \in f^{-1}(Q)$ then $T_p(f^{-1}(Q)) = (df_p)^{-1}(T_{f(p)} Q)$.

[Wortman, *Math 6510 Notes,* p. 66.]

17. [Sept. 28.] **Sphere is Simply Connected.** Prove that the sphere $\mathbb{S}^n$ is simply connected if $n > 1$. Let $p_0 \in \mathbb{S}^n$ be a fixed basepoint and $\gamma : \mathbb{S}^1 \to \mathbb{S}^n$ be a $\mathcal{C}^1$ map such that $\gamma(1) = p_0$. Show that there is a homotopy, a continuous map $H : \mathbb{S}^1 \times [0,1] \to \mathbb{S}^n$ such that $H(x,0) = \gamma(x)$ and $H(x,1) = p_0$ for all $x \in \mathbb{S}^1$ and $H(p_0, t) = p_0$ for all $t \in [0,1]$. Hint: if $n > 1$ show there is $y \in \mathbb{S}^n \backslash \gamma(\mathbb{S}^1)$ and use stereographic projection from y. [Guillemin & Pollack, *Differential Topology,* Prentice Hall, 1974, p. 45.]

18. [Oct. 1.] **Smooth Approximation Theorem.** Let M^m be a smooth manifold and $A, B \subset M$ be closed subsets. Let $f : M^m \to \mathbb{R}^n$ be continuous on M and smooth on A. Show:

(a) Given $\epsilon > 0$ there is continuous map $g : M \to \mathbb{R}^n$ which is smooth on $M - B$, is such that $f(x) = g(x)$ for all $x \in A \cup B$ and $|f(x) - g(x)| < \epsilon$ for all $x \in M$.

(b) f is homotopic to g relative to $A \cup B$ via an ϵ-small homotopy. That means there is a continuous map $h : M \times [0,1] \to \mathbb{R}^n$ such that $h(x,0) = f(x)$ and $h(x,1) = g(x)$ for all $x \in M$, such that $h(x,t) = f(x)$ for all $x \in A \cup B$ and $t \in [0,1]$ and such that $|f(x) - h(x,t)| < \epsilon$ for all $(x,t) \in M \times [0,1]$.

(c) Suppose that if $r : \mathbb{D}^n \to \mathbb{S}^{n-1}$ is a continuous retraction, a map such that $r(x) = x$ whenever $x \in \partial\mathbb{D}^n = \mathbb{S}^{n-1}$. Let $f : \mathbb{D}^n \to \mathbb{S}^{n-1}$ be a modification of r given by

$$f(x) = \begin{cases} r(2x), & \text{if } |x| \le \frac{1}{2}; \\[2ex] \dfrac{x}{|x|}, & \text{if } |x| > \frac{1}{2}. \end{cases}$$

Show that there is a smooth map $g : \mathbb{D}^n \to \mathbb{S}^{n-1}$ that is homotopic to r through maps to $\mathbb{S}^{n-1}$ and such that $f(x) = g(x)$ when $|x| \ge \frac{3}{4}$.

(d) Use Hirsch's argument to show that there is no continuous retraction $r : \mathbb{D}^n \to \mathbb{S}^{n-1}$.

[Bredon, *Topology and Geometry,* Springer, 1993, p. 96.]

19. [Oct. 3.] **Riemnannian Metric.** A *Riemannian Metric* for a smooth manifold M is a positive definite quadratic form $\langle\cdot,\cdot\rangle_x$ on T_xM defined for each $x \in M$ which varies smoothly from point to point: if $V(x)$ and $W(x)$ are any smooth vector fields on M then $x \mapsto \langle V(x), W(x)\rangle_x$ is a smooth function. Give two proofs that every smooth manifold M has a Riemannian metric.

20. [Oct. 5.] **Every Compact Manifold Embeds into Euclidean Space.** Let M^m be a smooth compact manifold. Show that for some large enough n there is a smooth embedding $f : M \to \mathbb{R}^n$. [John M. Lee, *Introduction to Smooth Manifolds, 2nd ed.*, Springer, 2013, p. 134.]

21. [Oct. 15.] **Veronese Surface.** Define maps $f : \mathbb{R}^3 \to \mathbb{R}^4$ and $g : \mathbb{R}^3 \to \mathbb{R}^5$ by

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} yz \\ xz \\ xy \\ x^2 - y^2 \end{pmatrix}, \qquad g\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} yz \\ xz \\ xy \\ \frac{1}{2}\left(x^2 - y^2\right) \\ \frac{1}{2\sqrt{3}}\left(x^2 + y^2 - 2z^2\right) \end{pmatrix}.$$

Viewing $\mathbb{RP}^2 = \mathbb{S}^2/\sim$ where $p \sim -p$, show that f and g descend to smooth embeddings $f : \mathbb{RP}^2 \to \mathbb{R}^4$, $g : \mathbb{RP}^2 \to \mathbb{R}^5$. In fact the embedding g, called the *Veronese Surface,* is *isometric* to $\mathbb{RP}^2$: if $\gamma : [0,1] \to \mathbb{RP}^2$ is a smooth curve, then $\operatorname{length}(\gamma) = \operatorname{length}(g \circ \gamma)$ where lengths are measured in the background Euclidean spaces.

[H. Brandsma, Ask a Topologist, Embedding of projective plane into $\mathbb{R}^4$, June 3, 2003, (`http://at.yorku.ca/cgi-bin/bbqa`) ; Q. Han & J.X. Hong, *Isometric Embeddings of Riemannian Manifolds in Euclidean Spaces,* American Mathematical Society, 2006, p. 39.]

22. [Oct. 17.] **Vector Bundles.**

    (a) The *Mŏbius Band*, $\mathcal{MB}$ is the space $[0,1] \times \mathbb{R}/\sim$ where $(0,x) \sim (1,-x)$. Show that $\mathcal{MB}$ is a smooth vector bundle.

    (b) Suppose that $\pi : E \to M$ is a smooth vector bundle of rank r. Show that E is trivial if and only if M has a *smooth global frame, i.e.,* smooth global sections $\sigma_1(p), \ldots, \sigma_r(p)$ such that $\{\sigma_1(p), \ldots, \sigma_r(p)\}$ is a basis for $\pi^{-1}(p)$ for each p.

    (c) Use (b) to determine whether $T\mathbb{S}^1$, $\mathcal{MB}$ or $T\Sigma_g$ are trivial.

    [Wortman, *Math 6510 Notes,* pp. 8–15, 85–93.]

23. [Oct. 19.] **Stability of Maps.** Prove that the Stability Theorem is false on noncompact domains. Let $\rho : \mathbb{R} \to \mathbb{R}$ be a smooth function with $\rho(s) = 1$ for $|s| < 1$ and $\rho(s) = 0$ for $|s| > 2$. Define the family of maps $f_t : \mathbb{R} \to \mathbb{R}$ given by $f_t(x) = x\rho(tx)$. Verify that this is a counterexample to all six parts, immersion, submersion, local diffeomorphism, transverse to $Q \subset N$ (take $Q = \{0\}$), embedding and diffeomorphism of the Stability Theorem [Wortman p. 80]. [Guillemin & Pollack, *Differential Topology,* Prentice Hall 1974, p. 38 and John M. Lee, *Introduction to Smooth Manifolds, 2nd ed.*, Springer, 2013, p. 149.]

24. [Oct. 22.] **Linear Vector Fields.** Let A be a real $n \times n$ matrix and define the matrix exponential by the series
$$e^A = \sum_{k=1}^{\infty} \frac{1}{k!} A^k.$$
which is convergent for all A.

    (a) Let $V(x) = Ax$ be a linear vector field on $\mathbb{R}^n$. Check that the flow generated by V is
    $$\phi_t(x_0) = e^{tA} x_0$$
    and that it is a one parameter group of diffeomorphisms.

    (b) If B is another real $n \times n$ matrix and $W(x) = Bx$ is its corresponding linear vector field, show that $[V, W]$ is a linear vector field and find its associated matrix. [Siavash Shahshahani, *An Introductory Course on Differentiable Manifolds,* Dover, 2016, p. 60.]

    (c) For these vector fields, compute the Lie derivative directly and check the formula
    $$\pounds_V W = [V, W].$$

25. [Oct 24.] **Point-Moving Diffeomorphism.** Let $B \subset \mathbb{R}^n$ be the open unit ball centered at the origin and let $y \in B$. Define a vector field on B and its resulting flow to find a diffeomorphism $f : B \to B$ such that $f(0) = y$, f fixes a neighborhood of ∂B and f is homotopic to the identity on B. Conclude that if M is smooth connected manifold and $p, q \in M$ then there is a diffeomorphism $F : M \to M$ such that $F(p) = q$ and F is homotopic to the identity on M. [University of Utah Preliminary Examination in Geometry/Topology, January 2017.]

26. [Oct. 26.] **Compact Perturbation of Complete Vector Field.** Let V be a smooth vector field on a smooth manifold M and assume that V has a flow that is defined on all of M and for all time. Let W be another vector field such that $W - V$ has compact support. Show that W has a flow that is defined on all of M and for all time. [University of Utah Preliminary Examination in Geometry/Topology, January 2014, (see also August 2015.).]

27. [Oct. 29.] **Euclidean Charts.** Let M^m be a smooth manifold and $X_1, X_2, \ldots, X_m$ be smooth vector fields on M that are linearly independent everywhere. Show that if $[X_i, X_j] = 0$ then about every point $q \in M$ there is a local coordinate system $x^1, \ldots, x^m$ such that $X_i = \frac{\partial}{\partial x^i}$, $i = 1, \ldots, m$. [S. Sternberg, *Lectures on Differential Geometry,* Chelsea, 1984, p.134.]

28. [Oct. 31.] **Foliation from Submersion.** Suppose M and N are smooth manifolds and $F : M \to N$ is a smooth submersion. Show that the connected components of the nonempty level sets of F form a foliation of M. [John M. Lee, *Introduction to Smooth Manifolds, 2nd ed.*, Springer, 2013, p. 513.]

29. [Nov. 2.] **Curve in a Submanifold.** Suppose M is a smooth manifold, $S \subset M$ is a smooth submanifold, Δ_p a smooth involutive k-plane distribution on M and $\gamma : (a, b) \to M$ a smooth curve in M.

    (a) Show that if $\gamma((a, b)) \subset S$ and S is an embedded submanifold then $\dot\gamma(t) \in T_{\gamma(t)}S$ for all $t \in (a, b)$.

    (b) Find a counterexample to (a) if S is not embedded.

    (c) Show that if $\gamma((a, b)) \subset S$ and S is an integral manifold of Δ_p then $\dot\gamma(t) \in T_{\gamma(t)}S$ for all $t \in (a, b)$.

    (d) Show that if $\dot\gamma(t) \in \Delta_{\gamma(t)}$ for all $t \in (a, b)$ then $\gamma((a, b))$ is containted in a single leaf of the foliation defined by Δ_p.

    [John M. Lee, *Introduction to Smooth Manifolds, 2nd ed.*, Springer 2013, pp. 124, 513.]

30. [Nov. 5.] **Abelian Lie Groups.** Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ such that $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$. Prove that G is Abelian. [University of Utah Preliminary Examination in Geometry/Topology, January 2017. See Wortman, *Math 6510 Notes,* p. 212.]

31. [Nov. 7.] **Subgroups of** $\mathrm{SL}_2(\mathbb{R})$**.** Let $U = \left\{\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right) : x \in \mathbb{R}\right\}$. Find all connected Lie subgroups o f $\mathrm{SL}_2(\mathbb{R})$ containing U. [University of Utah Preliminary Examination in Geometry/Topology, May 2017. See Wortman, *Math 6510 Notes,* p. 212.]

32. [Nov. 9.] **Surjectivity of exp.**

    (a) By showing that $\begin{pmatrix} -\frac{1}{4} & 0 \\ 0 & -4 \end{pmatrix}$ has no square root, prove that exp does not map the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ onto $\mathrm{SL}_2(\mathbb{R})$. [R. Bishop & R. Crittenden, *Geometry of Manifolds,* Academic Press, 1964, p. 33.]

    (b) Show that $\mathrm{SL}_2(\mathbb{R})$ is connected. [John M. Lee, *Introduction to Smooth Manifolds, 2nd ed.*, Springer, 2013, pp. 537, 563.]

33. [Nov. 12.] **Cartan's Lemma.** Let $\omega^1, \ldots, \omega^k$ be smooth one-forms on a smooth manifold M^n which are linearly independent at all points of M. Let $\alpha^1, \ldots, \alpha^k$ be smooth one-forms such that
    $$\textstyle\sum_{i=1}^k \alpha^i \wedge \omega^i = 0.$$
    Show that there are uniquely determined smooth functions ${c^i}_j = {c^j}_i$ such that
    $$\alpha^i = \textstyle\sum_{j=1}^k {c^i}_j\, \omega^j.$$
    [John M. Lee, *Introduction to Smooth Manifolds, 2nd ed.*, Springer, 2013, pp. 375.]

34. [Nov. 14.] Define an n-form on $\mathbb{R}^{n+1}$ by

$$\omega = \sum_{i=1}^{n+1} (-1)^i x^i \, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots dx^{n+1},$$

where "hat" means omit. Let $i : \mathbb{S}^n \to \mathbb{R}^{n+1}$ be the usual inclusion. Show that $i^*\omega$ is a non-vanishing n-form on $\mathbb{S}^n$, hence it is orientable. [John M. Lee, *Introduction to Smooth Manifolds, 2nd ed.*, Springer, 2013, pp. 375.]

35. [Nov. 16.] **Invariant Formula for Exterior Derivative and Involutivity.** Let ω be a smooth one form and let X, Y be smooth vector fields on the smooth manifold M^m. Then the invariant formula for the exterior derivative is

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]).$$

(a) Show that the invariant description of $d\omega$ agrees with the coordinate description.

(b) A smooth k-plane distribution on M may be defined using one forms. For every $p \in M$ there is an open neighborhood U of p in M and $m-k$ linearly independent one forms ω^i on U such that

$$\Delta_q = \left\{ X \in T_qM : \omega^{k+1}(X) = \omega^{k+2}(X) = \cdots = \omega^m(X) = 0 \right\}, \qquad \text{for all } q \in U.$$

Show that Δ_p is involutive if and only if there are one forms $\omega^i{}_j$ such that on U,

$$d\omega^i = \sum_{j=k+1}^{m} \omega^i{}_j \wedge \omega^j, \qquad \text{for all } i = k+1, \ldots, m$$

36. [Nov. 19.] **Orientability of the Tangent Bundle.** Let M be a smooth manifold. Show that the tangent bundle TM is orientable. [University of Utah Preliminary Examination in Geometry/Topology, Jan., 2016.]

37. [Nov. 21.] **Pullback by Homotopic Maps.** Let M and N be smooth compact oriented manifolds, both of dimension n. Suppose $f : M \to N$ and $g : M \to N$ are smoothly homotopic. Prove that for any $\omega \in \Omega^N(N)$, we have

$$\int_M f^*\,\omega = \int_M g^*\,\omega.$$

[University of Utah Preliminary Examination in Geometry/Topology, Jan., 2017.]

38. [Nov. 26.] **Integral of an Orientation Form.** Suppose that $U \in \mathbb{R}^n$ is an open set and $\psi_U \in \Omega^n(U)$ is an integrable, nonvanishing form that determines the orientation of U. Show that $\int_U \psi_U > 0$. Suppose that M^n is a smooth, compact, orientable manifold and $\omega \in \Omega^n(M)$ is an nonvanishing form that determines the orientation of M. Show that $\int_M \omega > 0$. [Wortman, *Math 6510 Notes*, p. 300.]

39. [Nov. 28.] **Divergence Theorem.** Let $M^3 \subset \mathbb{R}^3$ be a smooth, connected, compact imbedded submanifold with boundary. Let $V = v^i \frac{\partial}{\partial x^i}$ be a smooth vector field in the usual coordinates of $\mathbb{R}^3$.

(a) Find $\omega \in \Omega^2(\mathbb{R}^3)$ such that

$$d\omega = \operatorname{div}(V)\, dx^1 \wedge dx^2 \wedge dx^3, \qquad \text{where } \operatorname{div}(V) = \sum_{i=1}^{3} \frac{\partial v^i}{\partial x^i}.$$

(b) For $p \in \partial M$, let (u^1, u^2) be local coordinates for ∂M near p. Express $i^*\omega\left(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}\right)$ in terms of V, $i_*\left(\frac{\partial}{\partial u^1}\right)$ and $i_*\left(\frac{\partial}{\partial u^2}\right)$ where $i : \partial M \hookrightarrow M$.

(c) Deduce the usual Divergence Theorem using Stoke's Theorem where N is the outer unit normal and $d\sigma = \left|i_*\left(\frac{\partial}{\partial u^1}\right) \times i_*\left(\frac{\partial}{\partial u^2}\right)\right| du^1 \wedge du^2$ is the usual area form of ∂M

$$\int_M \operatorname{div}(V)\, dx^1 \wedge dx^2 \wedge dx^3 = \int_{\partial M} V \cdot N \, d\sigma.$$

[T. Aubin, *A Course in Differential Geometry,* Amer. Math. Soc., 2000, pp. 60–61.]

40. [Nov. 30.] **No Retraction.** Let M^m be a smooth compact oriented manifold with boundary.

(a) Let $f : \partial M \to \partial M$ be smooth a diffeomorphism. Show that f has no smooth extension to $F : M \to \partial M$. [Hint: Let $i : \partial M \hookrightarrow M$ so $f = F \circ i$. Consider $\int_{\partial M} f^*\psi_{\partial M}$.]

(b) Deduce that there is no smooth retraction of the Euclidean ball $F : \overline{B_1^n(0)} \to \mathbb{S}^{n-1}$ and conclude that any smooth map $g : \overline{B_1^n(0)} \to \overline{B_1^n(0)}$ has a fixed point.

[Guillemin & Pollack, *Differential Topology,* Prentice Hall, 1974, p. 186, and Siavash Shahshahani, *An Introductory Course on Differentiable Manifolds,* Dover 2016, p. 221.]

41. [Dec. 3.] **de Rham Cohomology of the Annulus.** Determine the de Rham cohomology of the annular region

$$A = \{(x, y) : 1 < x^2 + y^2 < 4\}.$$

[Frank Warner, *Foundations of Differentiable Manifolds and Lie Groups,* Scott, Foresman and Co., 1971, p. 158.]

42. [Dec. 5.] **Cohomology of Spheres.** Show $H^k_{dR}(\mathbb{S}^n) = \{0\}$ for $1 \le k < n$ by induction.

Hint: Suppose that the claim has been proved for all cases $j < k$ and $m < n$. Let $U^+ = \mathbb{S}^n - \{S\}$, $U^- = \mathbb{S}_n - \{N\}$ where N and S are north and south poles, respectively. Let $\alpha \in Z^k(\mathbb{S}^n)$. Let $\alpha^\pm$ be the restrictions of α to $U^\pm$.

(a) Show there are $\beta^\pm \in \Omega^{k-1}(U^\pm)$ such that $d\beta^\pm = \alpha^\pm$ and $\gamma \in \Omega^{k-2}(U^+ \cap U^-)$ such that $d\gamma = \beta^+ - \beta^-$.

(b) Let $\rho^\pm$ be a partition of unity subordinate to the cover $\{U^+, U^-\}$ such that $\rho^+ = 1$ in a neighborhood of N and $\rho^- = 1$ in a neighborhood of S. Let $\beta = \rho^+\beta^+ + \rho^-\beta^- + d\rho^+ \wedge \gamma$. Show that $\beta \in \Omega^{k-1}(\mathbb{S}^n)$ and $d\beta = \alpha$.

[Siavash Shahshahani, *An Introductory Course on Differentiable Manifolds,* Dover 2016, p. 247.]

The last day to turn in any remaining homework is Friday, Dec. 14.