MATH 5210-001

EXAM 1

Instructions. The exam is closed book/notes/calculator. There are 4 problems, each worth the same number of points. DO 3 OF THE 4 PROBLEMS. If you do more than 3, indicate which 3 you want graded. Justify your answers.

1a. State the definition of a norm on a vector space V, and the definition of two norms $\| \cdot \|_1$ and $\| \cdot \|_2$ being equivalent on V.

b. Show directly (without quoting a theorem) that the norms $\| \cdot \|_\infty$ and $\| \cdot \|_1$ (recall that $\|x\|_1 = |x_1| + \cdots |x_n|$, and $\|x\|_\infty = \sup |x_j|$) are equivalent on $\mathbf{R}^n$.

c. Show that on $C([0,1])$, the sup norm $\| \cdot \|_\infty$ and the L^1 norm, $\| \cdot \|_1$ are not equivalent.

2. State the definition of a set $K \subset X$ being compact, where X is a metric space. Are the following sets compact? Give a proof of your answer (you may use theorems proven in class).

a. $\overline{B}_1(0) \subset \mathbf{R}^n$.

b. $\overline{B}_1(0) \subset \ell^1$.

c. $\overline{A} \subset C([0, 1/2])$ with the sup norm, where $A = \{x^n : n = 1, 2, \ldots\}$.

3a. State the definition of $F : X \to Y$ is uniformly continuous on $A \subset X$, where X and Y are metric spaces.

b. Show that $T : C([a, b]) \to \mathbf{R}$ is uniformly continuous on $C([a, b])$ (with the sup norm) where T is defined as:

$$T(f) = \int_a^b f(x)\, dx$$

4a. Define what it means for a metric space X to be complete.

b. Let X be a compact metric space. Show that X is complete.