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A nonlocal curvature evolution problem in the plane

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Abstract. We consider Zhu's model for combustion in a Hele-Shaw cell which reduces to a nonlocal curvature evolution problem for a curve in the plane. The normal velocity is the sum of the curvature dependent burning rate and the fluid effects from the potential flow and gravity. We give the physical and geometric motivations and relate our study to various similar problems that have been considered. Due to the assumption of equal viscosities and zero surface tension, our model admits a relatively simple treatment. We give fingering criteria, geometric properties and stability results for the flow and its linearization about the rising circle solution.

In this section, we describe certain nonlocal geometric evolution problems currently of great interest in applied mathematics. The equations reduce to a curvature evolution problem for curves in the plane with nonlocal term. We shall describe two types of nonlocal problems that occur in potential flows and mention a few results in order to give context to our recent work about the stability of a combustion problem. Then we shall describe our model and some results. Details will appear elsewhere[Tr].

We formulate an evolution problem for closed curves in the plane. Let $X \in C^\infty(\mathbf{S}^1 \times [0,T), \mathbf{R}^2)$ be the position vectors of a one parameter family of embeddings of the circle. For given t, let $\Gamma_t = X(\mathbf{S}^1 \times \{t\}) \in \mathbf{R}^2$ be a smooth closed embedded curve and let Ω_1 and Ω_2 be the unbounded and bounded components of $\mathbf{R}^2 - \Gamma_t$. Let n denote the outward normal vector to Ω_2. We assume that the family of curves evolves according to

$$\begin{gathered} V = n \cdot \frac{dX}{dt} = \alpha(\kappa) + \mathcal{N}(\Gamma_t), \\ X(\theta, 0) = X(\theta), \end{gathered}$$

where V denotes the normal velocity and $X(\theta)$ a parameterization of the initial curve Γ_0, κ is curvature of Γ_t, $\kappa > 0$ for the circle, and α is a sufficiently smooth function. $\mathcal{N}$ is a nonlocal operator depending on the curve. We will discuss $\mathcal{N}$ of a particular type which occurs in potential flows. It is a geometric integral over the curve of powers of chords. More general $\mathcal{N}$ occur sometimes, for example to represent the effect of more complicated fluid behavior, *e.g.* Stokes flow, away from the interface [Pz].

With $\mathcal{N} \equiv 0$, such flows have been suggested as models for physical phenomina. $\mathcal{N}$ takes into account nonlocal effects *e.g.* fluid motion away from Γ, pressure influenced by surface tension and body forces such as gravity or suction. Examples are the motion of a phase boundary for perfect conductors $\alpha = a\kappa + b$, [AG] or crystal growth[S] or combustion $\alpha = ae^{-b\kappa}$ [M] . However, a more complete job modelling physical phenomena is done when $\mathcal{N}$ included. In flame propogation [JS], [PT], the nonlocal term reflects flow around the flame. Such nonlocal terms model of oil recovery - flow through porous medium[BK], [H2], injection molding into a Riemannian surface[VE], tumor growth[BC], Ostwald ripening[MV], vortex patch motion for inviscid fluids[CT]. There are extensive bibliographies [V], [Hw], [G].

There is a fairly complete theory for purely curvature driven flows ($\mathcal{N} \equiv 0$). For inward flow, $\alpha = -\kappa$,

Theorem. [HG], [Gr] *Let Γ_0 be a smooth embedded curve. There is a time T so that Γ_t evolves through embedded curves for $0 \le t < T$. It becomes convex and collapses to a round point as $t \to T-$. i.e., there is a function $f(t) \to \infty$ as $t \to T$ so that $f(t)\Gamma_t$ tends to the unit circle in $C^{2+\delta}$ at $t \to T$.*

For outward flow and general α the flow may not preserve embeddedness. However, there is good theory if the curvature of the initial curve is not too negatively curved.

Theorem. [CL] *Suppose α is a smooth function such that*

$$\alpha(\kappa) > 0,\ \alpha'(\kappa) < 0 \text{ for all } \kappa \text{ and } \alpha(\kappa) \to \infty \text{ as } \kappa \to -\infty.$$

Let Γ_0 be a smooth embedded curve such that $\int_I \kappa\, ds > -\pi$ for all intervals $I \subset \Gamma_0$. Then there exists a solution $X \in C^\infty(\mathbf{S}^1 \times [0,\infty))$ to $V = \alpha(\kappa)$, with initial curve Γ_0 such that all the curves Γ_t remain embedded, for every point $p \in \mathbf{R}^2$ there is a critical time T_1 so Γ_t is star shaped with respect to p if $t \ge T_1$ and a second time $T_2 \ge T_1$ so that Γ_t is convex if $t \ge T_2$. Moreover, there is a function $f(t) \to 0$ as $t \to \infty$ so that $f(t)\Gamma_t$ tends to the unit circle in $C^{2+\delta}$ as $t \to \infty$.

We mention only a few higher dimensional generalizations. The first geometer to consider curvature flows seems to be Firey[F] who studied hypersurfaces evolving by Gauss-Kronecker curvature. He imagined the process describes a stone wearing by random angle collisions with a riverbed. Mean curvature flow has been extensively studied [B], [Hu], [ES1], [CG], [I]. Gauss-Kronecker and more general flows have also been studied [To], [Cw], [CN], [A].

Nonlinear term: flows in a Hele-Shaw cell. Two immiscible fluids occupy the narrow gap between two parallel glass plates. The interface isidealized as a curve $\Gamma_t \subset \mathbf{R}^2$. The drag along the plates is felt by the entire fluid. Averaging the Navier Stokes flow over the gap gives Darcy's law. $p_i(x,y)$ is the pressure in

Ω_i, $c_i = -\ell^2/12\mu_i$, ℓ is the gap width and μ_i is the viscosity. In $\mathbf{R}^2 - \Gamma_t$, the fluid satisfies

$$\begin{aligned} w_i &= c_i \nabla p_i && \text{Darcy's Law} \\ p_1 &\to F(z) && \text{as } z \to \infty. \text{ (far field condition)} \\ \text{div } w_i &= 0 && \text{Incompressibility.} \end{aligned}$$

where F is driving pressure or sources and sinks. The constant T determines the surface tension. On Γ_t,

$$\begin{aligned} [p] =& T\kappa && \text{Surface tension.} \\ [n \cdot w] =& 0 && \text{Kinematic condition.} \\ \mathcal{N}(\Gamma_t) =& n \cdot w && \text{Interface carried by flow.} \end{aligned}$$

For $z \in \Gamma_t$ the jump along the interface

$$[f](z) = \lim_{w \in \Omega_1,\ w \to z} f(w) - \lim_{w \in \Omega_2,\ w \to z} f(w).$$

We indicate why the problem is more difficult if $T > 0$. Along the interface the kinematic normal velocities condition says that Γ_t is a material curve which follows the fluid motion. We can represent $n \cdot w_1 = \mathcal{K}\gamma$ as an operator of the *vortex sheet strength* $\gamma = [\mathbf{t} \cdot w]$ and $\mathbf{t} = in = z_s/|z_s|$. In fact, $\overline{w_i}$ is analytic in $z = x + iy \in \mathbf{C} - \Gamma_t$, and is given by the Cauchy formula

$$\bar{w}(z) = \frac{1}{2\pi i} PV \int_\Gamma \frac{\gamma(z')\bar{\mathbf{t}}(z').dz'}{z' - z}.$$

On Γ_t, by the Plemelj formula, $[\bar{w}] = \gamma\bar{\mathbf{t}}$ and $(\bar{w}_1 + \bar{w}_2) = 2\mathcal{K}\gamma$. Also $[\mathbf{t} \cdot \nabla p] = [p_s] = T\kappa_s$, yielding a singular integral equation for γ

$$\gamma = \frac{4c_1c_2}{c_1 + c_2}\mathcal{K}\gamma + \frac{c_1 + c_2}{c_1 - c_2}T\kappa_s$$

thus, $\mathcal{N}(\Gamma_t) = c_3 T\bar{n} \cdot \mathcal{K}(I - c_4\mathcal{K})^{-1}\kappa_s$ which is a third order pseudo-differential operator. Procedures to handle such problems well numerically have been developerd only recently[HL]. The computational results compare well to experiments (*e.g.* [P] where an air bubble blown into a horizontal Hele-Shaw cell filled with glycerine which develops unstable fingers).

In case there is a large viscosity contrast, as with oil and water, or water and air, where viscosities differ by a factor of ten, the model is usually simplified to constant pressure in the less viscous fluid, the so-called *one phase* model.

Mullins-Sekerka problem. The fluids problem is closely related to the problem for evolving phase boundaries which was proposed by Mullins-Sekerka[MS]. The one phase versions reduce to the same problem. The boundary condition at the interface becomes a Dirichlet condition and the velocity is now given by a jump, therefore this problem may also have geometric interest[VE]. Suppose $\Omega_2 \subset\subset \Omega_0 \subset\subset \mathbf{R}^2$. Ω_2 represent the solid and $\Omega_1 = \Omega_0 - \overline{\Omega_1}$ the liquid phases of a substance in a smooth tank; u_i their temperature and a, b, T constants. The liquid temperature is brought below the freezing temperature. Starting from a frozen seed, the solid phase $\Omega_2(t)$ grows, being driven by the undercooled liquid.

$$\begin{aligned} \Delta u =&0 && \text{in } \Omega_0 - \Gamma. \textit{ (or the heat equation in the Stefan problem)} \\ \partial_n u =&0 && \text{on } \partial\Omega_0 \\ u =& -T\kappa && \text{on } \Gamma_t. \textit{ (Gibbs-Thomson law)} \\ V =& \left[\frac{\partial u}{\partial n}\right] && \text{on } \Gamma_t \end{aligned}$$

This model is sometimes also called the two phase Stefan problem with Gibbs-Thomson Law for melting.

We state some existence and stability results for the Mullins-Sekerka problem.

Theorem. [C], [CH], [ES] *Let $\Gamma_0 \in C^{2+\alpha}$. There exists a $t_1 > 0$ so that the Mullins Sekerka problem has a unique solution in $C^{2+\alpha,1+\alpha/2}$ which is smooth for $t > 0$. For $k \in \mathbf{N}$, if Γ_0 is close enough to the round sphere in $C^{2+\alpha}$ then the solution approaches the round sphere exponentially fast in the C^k. (The theorem holds for $\mathbf{R}^n$ where κ is the mean curvature of Γ_t)*

Higher dimensional existence of global weak solutions[AW], [Lu] and short time existence[R] are known for the Stefan problem with Gibbs-Thomson law. In two dimensions for the one phase problem, short time existence and long time existence and asymptotic circularity is known small perturbations of the circular bubble[CP]. Existence and stability for unbounded geometry is known[DR]. The $T = 0$ case has also been studied, *e.g.* for oil recovery [VE], [T], [NT].

Chemical reaction problem. We restate the particular problem under consideration. We seek a family of closed curves $\{\Gamma_t\}_{t\geq 0} \subset \mathbf{R}^2$ satisfying a curvature evolution with nonlocal term.

$$V = n \cdot \frac{dX}{dt} = \alpha(\kappa) + \mathcal{N}(\Gamma_t)$$

where Γ_0 is given embedded closed curve. n is the outer normal and $\kappa > 0$ for circle. We present Zhu's model for a chemical reaction along a front (which we'll call flame propogation) in a vertical Hele-Shaw cell[Z]. The model is designed to study buoyancy effects on the shapes of flames. Assume Ω_1, Ω_2 are components of $\mathbf{R}^2 - \Gamma$, Ω_2 is bounded. Assume Ω_1 is *fuel* and Ω_2 is the *combustion product* which is lighter than the fuel. The densities satisfy $\rho_1 > \rho_2$. We assume viscosities *are equal* $\mu_1 = \mu_2 = \mu$. We assume that the chemicals are miscible, there is no surface tension along the interface, $T = 0$. But, the rate of reaction along the interface is given by $\alpha(\kappa)$ where $\alpha \in C^1$, $\alpha > 0$, $\alpha' < 0$. A typical $\alpha = S_1 e^{-b\kappa}$. S_1 is the burning rate of a linear flame. The fire burns outward. Experiments with an aqueous hydrosulfite-iodate autocatalytic reaction in a vertical Hele-Shaw cell, in which the density varies about .02% due to temperature differences, show that the combustion product grows unstable fingers on the upper side, where the heavier fuel lies above the lighter product[AR]. Numerical experiments[Z] evolve unstable fingers in agreement with the experiment, whenever the fluid effects are strong relative to the combustion effects. However, when the combustion effects dominate, small perturbations of a circular product region die out. Our analysis confirms these observations.

Let $p(x,y)$ be the pressure and y the vertical coordinate function.

$$\begin{aligned}
w_i =& \frac{-\ell^2}{12\mu_i}\nabla\left(p_i - g\rho_j y\right) && \text{in } \Omega_i \\
\Delta p_i =& 0 && \text{on } \Omega_i \\
p_1 \to & -g\rho_1 y && \text{as } z \to \infty \\
[p] =& p_2 - p_1 = T\kappa && \text{on } \Gamma_t, \\
[n\cdot w] =& 0 \\
\mathcal{N}(\Gamma_t) =& n\cdot w_i
\end{aligned}$$

We assume J.Y. Zhu's simplifying hypothesis: $\mu_1 = \mu_2 = \mu$ and $T = 0$. Writing $n_2 = n\cdot(0,1)$,

$$0 = [n\cdot w] = -\frac{\ell^2}{12\mu}\left(\left[\frac{\partial p}{\partial n}\right] - g[\rho]n_2\right)$$

thus

$$\left[\frac{\partial p}{\partial n}\right] = g[\rho]n_2.$$

Thus the pressure is exactly given by the single layer potential.

$$p_i(z) = -g\rho_1 y + \frac{g[\rho]}{2\pi}\int_{\Gamma_t}\log|z-z'|\,n_2(z')\,ds(z').$$

Take $\partial/\partial n$ and use Plemelj's formula,

$$\mathcal{N}[\Gamma_t](z) = -\frac{g[\rho]\ell^2}{24\mu}n_2(z) - \frac{g[\rho]\ell^2}{24\pi\mu}PV\!\!\int_{\Gamma_t} n(z)\cdot\frac{z-z'}{|z-z'|^2}n_2(z')ds(z').$$

For C^2 curves, since as $z'\to z$,

$$n(z)\cdot\frac{z-z'}{|z-z'|}\to 0, \qquad n(z)\cdot\frac{z-z'}{|z-z'|^2}\to\frac{\kappa(z)}{2}$$

the integral is not singular.

Let $z(\theta,t) = R(\theta,t)e^{i\theta} + c_1 it + z_0$ where $c_1 = -\frac{g[\rho]\ell^2}{24\mu}$. Then the vertical translations cancel out and the radius function R satisfies

$$\begin{aligned}
RR_t =& \alpha\left(\frac{-RR_{\theta\theta} + 2R_\theta^2 + R^2}{(R^2+R_\theta^2)^{3/2}}\right)(R^2+R_\theta^2)^{1/2} + \\
&+\frac{c_1}{\pi}\int_{-\pi}^{\pi}\frac{\begin{pmatrix} R(\theta,t)^2 - R(\theta,t)R(\eta,t)\cos(\theta-\eta) \\ -R_\theta(\theta,t)R(\eta,t)\sin(\theta-\eta)\end{pmatrix}\begin{pmatrix} R(\eta,t)\sin\eta - \\ R_\theta(\eta,t)\cos\eta\end{pmatrix}d\eta}{R(\theta,t)^2 - 2R(\theta,t)R(\eta,t)\cos(\theta-\eta) + R(\eta,t)^2}
\end{aligned}$$

One solution is a growing rising circle $R(\theta,t)=\mathcal{R}(t)$ where

$$\mathcal{R}'(t)=\alpha\left(\frac{1}{\mathcal{R}}\right)=:\beta(\mathcal{R}),\qquad \mathcal{R}(0)=R_0.$$

β is a convenient version of α satisfying $\beta'>0$ and $S_0\le\beta(\mathcal{R})\le S_1$ for $\mathcal{R}>0$. Is it stable? Do small perturbations of $\mathcal{R}$ converge to $\mathcal{R}$ at least in some parameter regime? *i.e.* does $R(\theta,t)/\mathcal{R}(t)\to 1$ if $R(\theta,0)\approx R_0$? Is $\mathcal{R}$ linearly stable? Which modes grow at $t=0$, *i.e.* how many fingers form? Does every solution become convex?

To study these questions, we find that the linearization around $\mathcal{R}$ takes the following form. Substituting $R(\theta,t)=\mathcal{R}(t)+\varepsilon u(\theta,t)$ and collecting ε^1 terms yields

$$\text{(L)}\qquad \frac{\partial u}{\partial t}=\beta'(\mathcal{R})(u_{\theta\theta}+u)+\frac{c_1}{\mathcal{R}}\mathcal{M}u$$

where $0<\beta\in C^1$ and $\beta'>0$, and

$$\mathcal{M}u=\frac{\partial}{\partial\theta}\Big(\mathcal{H}[u]\sin\theta+(u-u_0)\cos\theta\Big)$$

where $\mathcal{H}u$ is the Hilbert transform

$$\mathcal{H}[u](\theta)=\frac{1}{2\pi}PV\!\!\int_{-\pi}^{\pi}u(\theta-\sigma)\cot\left(\frac{\theta}{2}\right)\,d\sigma.$$

The idea to solve the linearized equation is to regard the nonlocal term as a perturbation of the heat equation.

Theorem. *Choose $k\in\mathbf{N}$ and $t_1>0$. Let $\beta\in C^{k+\delta}[0,\infty)$ and $\phi\in C^{2+k+\delta}(\mathbf{S}^1)$. Then there is a unique solution to* (L) *and $u(\cdot,0)=\phi$ for all $t\ge 0$. The solution satisfies*

$$|u|_{C^{k+\delta,(k+\delta)/2}(\mathbf{S}^1\times[0,t_1])}\le c(\beta,\delta,|\phi|_{k+\delta},t_1).$$

Idea of proof. Energy etimates are used to provide H^1 bounds for all t. $\mathcal{M}$ is a first order perturbation. By the theorem of Privalov, $|\mathcal{H}u|_{k+\delta}\le c|u|_{k+\delta}$. Also, for functions depending on a parameter, an easy argument for any $\varepsilon>0$ gives an estimates of the parabolic Hölder norms, $|\mathcal{H}u|_{k+\delta-\varepsilon,(k+\delta-\varepsilon)/2}\le c|u|_{k+\delta,(k+\delta)/2}$. Since the perturbation is first order, there is room to maneuver and this suffices. The existence on the interval $[0,t_1]$ follows from standard estimates for the heat equation and an application of the Shauder fixed point theorem to the map which takes $w\in C^{+\delta,(1+\delta)/2}$ to the solution of

$$\begin{aligned}u_t-\beta'(\mathcal{R})(u_{\theta\theta}+u)&=c_1\mathcal{M}w/\mathcal{R}\\ u(0,\theta)&=u_0(\theta).\end{aligned}$$

By Privalov's estimate and interpolating the right side shows that the $c^{1+\delta,(1+\delta)/2}$ norm of any fixed point in $[0,t_1]$ is bounded by the H^1 norm, which in turn is bounded by the energy estimate. □

However, this argument does not give a good enough explicit time dependence on the solution to prove stability. The energy inequality only says that the solution grows lik $\mathcal{R}(t)$.

Poincaré's method. Further information, such as the evolution of individual Fourier modes of the initial data is needed to prove properties of the solution and substitute for the dispersion analysis. Poincaré studied tides in the ocean by reducing the resulting oblique boundary value problem for harmonic functions in the disk to a corresponding integral equation for holomorphic function on the boundary[Mu]. In this case the problem reduces to a linear complex differential equation in the disk, which can be studied by elementary means.

It is convenient to change time to

$$\tau = \int_0^t \frac{dt}{\mathcal{R}(t)}.$$

Let U be a harmonic function on the unit disk so that $U(e^{i\theta}) = u(e^{i\theta})$ and V its harmonic conjugate with $V(0) = 0$. Let $\Phi = U + iV$. Then $\mathcal{H}u = V|_{\mathbf{S}^1}$. (L) becomes on $\mathbf{S}^1$

$$0 = \Re\left(-\frac{\partial\Phi}{\partial\tau} - \frac{\beta_\tau}{\beta}\left(z^2\Phi'' + z\Phi - \Phi\right) + ic_1\left(\Phi' - \frac{\Phi - \Phi(0)}{z}\right)\right)$$

where "$'$" $= d/dz$. $\Phi(\cdot,\tau)$ is analytic. Assuming it extends over $|z| \leq 1$, then

$$\frac{\partial\Phi}{\partial\tau} = -\frac{\beta_\tau}{\beta}\left(z^2\Phi'' + z\Phi - \Phi\right) + ic_1\left(\Phi' - \frac{\Phi - \Phi(0)}{z}\right)$$

so the problem becomes complex valued but local on the unit disk. Then coefficients $\Phi(z,\tau) = \sum_0^\infty a_k(\tau) i^k z^k$ satisfy,

$$\text{(R)} \qquad \frac{\partial}{\partial\tau}\left(\beta^{k^2-1} a_k\right) = c_1 k \beta^{k^2-1} a_{k+1}$$

which can be explicitly integrated and estimated. We state some consequences of this recursion formula.

(1) Energy cascade is from high modes to lower modes. Since $\beta' > 0$, this is a manifestation of the fact that the curvature terms regularize (L).
(2) Polynomial initial data have polynomial solutions. If Φ_n is the solution of the initial value problem with $\Phi_n(z,0) = z^n$ then Φ_n is a polynomial of degree n whose lowest order z^1 coefficient grows at the fastest rate, like τ^{n-1}.
(3) If $\alpha \in C^1[0,T]$ then series converges on $[0,T]$. In particular, this implies a second existence proof.
(4) $|\Phi(\tau)|_0 = \mathbf{o}(e^{\varepsilon\tau})$ as $\tau \to \infty$ for all $\varepsilon > 0$. This implies that $R/\mathcal{R}$ has *linearized stability* since $\mathcal{R}$ grows linearly in t (hence exponentially in τ.)
(5) If β constant and the combustion rate doesn't depend on curvature, then there is still local existence, say for analytic initial data. However, blowup can happen in an arbitrarily short time for arbitrarily small perturbations of the initial data. In this case the first order PDE can be integrated to show that soltuions takes the form $\Phi(z) = a + zf(z + i\tau)$ where f is an arbitrary function analytic in a neighborhood of the unit disk. The $T = 0$ Hele-Shaw problem can be integrated similarly[HH] and develops singularities[Ta]. The model vorticity equation behaves similarly[CM].

(6) $\Phi(z,0)$ polynomial then $\Phi(\mathbf{S}^1,\tau)$ is convex for τ sufficiently large. In other words, *fingers coalesce* by the flow. The same statement holds for initial data which is close enough to polynomial. The downward cascade simply means that the lowest mode eventually swamps all the others.

(7) More fingers grow initially if c_1 large. To estimate the number of fingers that form initially, we compute the relative growth of the sup norm of the n-th mode at the initial time.

$$\left.\frac{\partial \log |\Phi_n|_0}{\partial \log \mathcal{R}}\right|_{\tau=0} > 1 \quad \text{if } 2 \le n \le 1 + \frac{c_1\beta(R_0)}{\beta_\tau(R_0)}$$

Hence fingers tend to form in more lower modes if the regularizing combustion effects are weak compared to the fluid effects. This agrees with the dispersion analysis of linear flame fronts[Z].

(8) If α regular at zero, u is analytic in θ for $t > 0$ and satisfies a Gevrey-like estimate. Let $S_0 = \beta(R_0)$. There are c_2, c_3 so

$$\left\| \left(\frac{\beta(\mathcal{R}(t))}{S_0}\right)^{2\Delta} u \right\|^2 = \mathcal{E}u \le c_2 \|\phi\|_{L^2(\mathbf{S}^1)} t^{c_3} \qquad \text{if } t \ge 1.$$

In terms of series, if $u = \sum a_k(t)e^{ik\theta}$ then $\mathcal{E}u = \sum (\beta/S_0)^{2k^2}|a_k|^2$. Since $\beta/S_0 > 1$, boundedness of $\mathcal{E}u$ implies exponential decay of Fourier coefficients in k which implies analyticity. Using (R), this is proved by finding an energy estimate for $\mathcal{E}u(\tau)$ where u is any solution. Since the polynomial solutions form a basis, the energy estimate is established using a Galerkin procedure.

The presence of the combustion term depending on curvature makes the equation regular as does the curvature term in case $T > 0$ withiout combustion. However, as in that case, β constant case cannot be viewed as being regularized by solutions of $0 < \beta' \to 0$ [Ta]. For example, long time existence holds for $\beta' > 0$ but completely fails when β is constant.

The rising circle solution of the nonlinear equation is relatively stable. We show that the solution $\mathcal{R}$ of the nonlinear

$$V = \alpha(\kappa) + c_1\mathcal{N}(\Gamma_t) \tag{N}$$

is relatively stable. *i.e.* if $R(\theta,0)/\mathcal{R}(0) \approx 1$ then solution exists for all time and $R/\mathcal{R} \to 1$ as $\tau \to \infty$. Following [TW] we parameterize

$$R(\theta,\tau) = \mathcal{R}(\tau)e^{u(\theta,\tau)}$$

and write κ for the curvature of $\theta \mapsto e^{u+i\theta}$. The equation becomes

$$u_\tau = \alpha\left(\frac{-u_{\theta\theta} + u_\theta^2 + 1}{\mathcal{R}e^u(1+u+\theta^2)^{3/2}}\right)e^{-u}\sqrt{1+u_\theta^2} - \alpha\left(\frac{1}{\mathcal{R}}\right) + c_1\mathcal{N}[u] \tag{E}$$

where the modified $\mathcal{N}[u]$ becomes

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K(\theta.\eta,\tau)\frac{\partial}{\partial\eta}\Big[e^{u(\eta,\tau)-u(\theta,\tau)}\cos(\eta)\Big]\,d\eta$$

where

$$K(\theta,\eta,\tau) = \frac{\sinh\big(u(\eta,\tau)-u(\theta,\tau)\big) - u_\theta(\theta,\tau)\sin(\eta-\theta)}{\cosh\big(u(\eta,\tau)-u(\theta,\tau)\big) - \cos(\eta-\theta)}.$$

The equation can be solved using a similar procedure to that used in the linear equation. Let $Q := \mathbf{S}^1 \times [0,\infty)$,

$$X^{2+\delta}(S) := C^{2+\delta,1+\delta/2}(S)$$
$$X^{2+\delta}_{k_2}(Q) := \left\{ v \in X^{\delta+2}_{\mathrm{loc}}(Q) : \sup_{\tau\geq 0} e^{k_2\tau}|v|_{X^{2+\delta}(\mathbf{S}^1\times\{\tau\})} < \infty \right\}$$

be time weighted Hölder spaces. The idea is to view the nonlinear term as a perturbation of curvature flow. One checks that by choosing the initial data and the fluids constant c_1 sufficiently small that one can arrange that solutions of the nonlinear equation (fixed points of the solution map) remain in a fixed ball of X_{k_2} independent of the time interval $[0,t_1]$ in which solutions are constructed.

Theorem. *Suppose $\beta \in C^{5+\delta}$ satisfies $\beta' > 0$, $s \le \beta/S_0 \le s + (1-s)e^{-k_1\tau}$ where $s = S_0/S_1 < 1$, $|\log\beta|_5 \le c_2 e^{-k_1\tau}$. There are constants c_i depending on c_2, k_1, S_0, S_1, R_0 so that if $\delta \le c_3$, $|u_0|_{2+\delta} \le c_4$, $c_1 \le c_5$ and $k_2 \le c_6$ then there is a solution $u \in X^{2+\delta}_{k_2}$ of* (E) *so that $u(\theta,0) = u_0(\theta)$. Moreover, for all $l \in \mathbf{N}$ there is c_7 also depending on $l \ge 2$ so that the solution $u \in X^{l+\delta}_{k_2}$ so that for $\tau > 1/l$,*

$$|u|_{X^{l+\delta}(\mathbf{S}^1\times\{\tau\})} \le c_7\left(|u_0|_{2+\delta}e^{-S_1\tau} + c_1 e^{-k_2\tau}\right).$$

Idea of the proof. View the nonlinear term as a perturbation of the outward mean curvature flow ($\mathcal{N} = 0$). Set up a fixed point argument in $X^{2+\delta}_{k_2}(\mathbf{S}^1\times[0,t_1])$. Apriori estimates for solutions to

$$u_\tau = \alpha(\kappa/\mathcal{R}) - \beta(\mathcal{R}) + f, \qquad \text{and } u(0) = u_0$$

where $f = c_1\mathcal{N}w$ for $w \in X^{2+\delta}_{k_2}$ and estimates of the nonlocal tern show that $u \in X^{2+\delta}_{k_2}$. A bootstrapping argument gives compactness. Then argue that for $|u_0|_{2+\delta}$ and $|c_1|$ small enough but *independent of* t_1, any fixed point (solution of the nonlinear equation) is in a uniform ball in $X^{2+\delta}_{k_2}$. □

The argument depends some lemmata. The first says that the nonlocal term can be decomposed as a Hilbert transform plus a more regular operator, and thus satisfies estimates similar to the linear case.

Lemma. *For every $\delta > \varepsilon > 0$, $k, j \in \mathbf{Z}_+$ there is a c_8 so that if $D = \partial^j_\theta\partial^k_\tau$ and $Du \in C^{1+\delta,(1+\delta)/2}$ then $D\mathcal{N}u \in C^{\varepsilon,(1+\varepsilon)/2}$ so that*

$$|D\mathcal{N}u|_{\varepsilon,(1+\varepsilon)/2} \le c_8|Du|_{1+\delta,(1+\delta)/2}$$

c_8 depends on ε, k, $|u_\theta|_0$ and $|Du|_{1+\delta,(1+\delta)/2}$.

Idea of the proof. One can decompose the kernel so that and $K = f + g\cot((\theta - \eta)/2)$ where f, g have better regularity. Thus $\mathcal{N}u = \int fh\,dx + \mathcal{H}(ghu)$ where $h = (e^{u(\eta,\tau)-u(\theta,\tau)}\cos\eta)_\eta$. For example, one term is

$$\frac{\cosh(u(\eta,\tau)-u(\theta,\tau)) - 1}{1-\cos(\eta-\theta)}$$

which behaves like u_θ^2. □

The loss of smoothness with respect to the time variable or with respect to any parameter is as for the Hilbert transform.

The second is to find apriori estimates for the inhomogeneous mean curvature equation. The essential point is that if the right side comes from an element of X_{k_2} and decays exponentially, then long time estimates can be found for solutions that depend only on the initial data and right side but not on the time interval $[0, t_1]$. Estimates for derivatives are very similar. We illustrate by presenting just a one sided C^0 estimate. The equation is

$$u_\tau = \alpha\left(\frac{\kappa}{\mathcal{R}}\right) - \beta(\mathcal{R}) + f(\theta,\tau) \tag{IMC}$$

where we have written κ for the curvature of $e^{u(\theta,\tau)}e^{i\theta}$ for convenience.

Lemma. *Suppose $\beta \in C^1$ such that $\beta' > 0$, $\beta(R_0) = S_0 > 0$, $s \le S_0/\beta \le s + (1-s)e^{-k_1\tau}$, $s = S_0/S_1$, $|(\log\beta)_\tau| \le c_4 e^{-k_1\tau}$. Suppose that $f \in C$ satisfies $|f| \le c_5 e^{-k_2\tau}$ where $k_2, c_5 < S_1$. Then there is a constant c_9 so depending on $S_0, S_1, k_1, c_4, |f|_0, |u_0|_0$ so any $C^{2,1}$ solution u of* (IMC) *satisfies*

$$e^u - 1 \le (e^{|u_0|_0} - 1)e^{-S_1\tau + (c_4+S_0)\min\{\tau, 1/k_1\}} + c_5 c_9 \frac{e^{-k_2\tau} - e^{-S_1\tau}}{S_1 - k_2}.$$

Idea of the proof. First find constant bounds using the maximum principle. The idea is to construct a supersolution $M(\tau)$ as the solution to an ODE which depends only on τ. Then to use finer inequlities to find sharper upper solutions.

Provided $M(0) \ge |u_0|_0$ we suppose there is a first point (θ_0, τ_0) where $u = M$. There $u_\theta = 0$ and $u_{\theta\theta} \le 0$ and $\kappa \ge e^{-M} > 0$. This says that we're within the range to substitute to β. Thus at the maximum point,

$$u_\tau - e^{-u}\beta\left(\frac{\mathcal{R}}{\kappa}\right) = -\beta(\mathcal{R}) + f$$

$$u_\tau = S_1 e^{-M} + |f|_0 - \frac{S_1}{1 + (S-1)e^{-k_1\tau}} = M_\tau.$$

Thus if $|f|_0 < S_1$ then the solution of the last ODE assumed to be satisfied by M (which is linear in e^M) is bounded by c_{10}.

A more refined estimate follows since $\beta(\mathcal{R}e^{-M})$ has a better upper bound. Using derivative assumptions on β and Taylor's formula

$$\begin{aligned} e^M M_\tau &\le \beta(\mathcal{R}) - \beta(\mathcal{R})e^M + fe^M + c_4 e^{-(1+k_1/S_1)M - k_1\tau}(e^M - 1), \\ (e^M - 1)_\tau &\le (-\beta(\mathcal{R}) + c_4 e^{-k_1\tau})(e^M - 1) + c_5 e^{-k_2\tau}e^M, \\ &\le (-S_1 + (c_4 + S_0)e^{-k_1\tau})(e^M - 1) + c_5 c_{10} e^{-k_2\tau}. \end{aligned}$$

Thus the integral grows until the sign of the leading term becomes negative. Then the linear term and inhomogeneous term decay. The esimate follows by integrating the new differential inequality satisfied by the refined M. □

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