$\boxed{\text{Solutions to Exercises in } \textit{Theory of Ordinary Differential Equations}}$

**1.** Using the definition of $f$, the definition of $y$, the definition of partial derivative, and the second enumerated property of flows, we have

$$\begin{aligned} f(y(t)) &= \left.\frac{\partial\varphi(s, y(t))}{\partial s}\right|_{s=0} = \lim_{h\to 0}\frac{\varphi(h, y(t)) - \varphi(0, y(t))}{h} \\ &= \lim_{h\to 0}\frac{\varphi(h, \varphi(t, x_0)) - \varphi(0, \varphi(t, x_0))}{h} = \lim_{h\to 0}\frac{\varphi(t+h, x_0) - \varphi(t, x_0)}{h} \\ &= \frac{\partial\varphi(t, x_0)}{\partial t} = \dot{y}(t). \end{aligned}$$

The definition of $y$ and the first enumerated property of flows tell us that $y(0) = \varphi(0, x_0) = x_0$.

**2.** There are uncountably many solutions of the given IVP. Separation of variables shows that for each interval $\mathcal{I}$ on which a solution $x(t)$ is positive there is a constant $c \le \inf\mathcal{I}$ such that $x(t) = (t-c)^2$ for all $t \in \mathcal{I}$. Similarly, if $x(t) < 0$ throughout $\mathcal{I}$, then $x(t) = -(t-c)^2$ on $\mathcal{I}$ for some $c \ge \sup\mathcal{I}$. These two facts imply that any solution of the IVP is of one of the following 4 forms:

$$\begin{aligned} x(t) &:= 0, \\ x(t) &:= \begin{cases} -(t-a)^2 & \text{if } t < a \\ 0 & \text{if } a \le t \end{cases} && (a \le 0) \\ x(t) &:= \begin{cases} 0 & \text{if } t \le b \\ (t-b)^2 & \text{if } b < t \end{cases} && (0 \le b) \\ x(t) &:= \begin{cases} -(t-a)^2 & \text{if } t < a \\ 0 & \text{if } a \le t \le b \\ (t-b)^2 & \text{if } b < t \end{cases}, && (a \le 0 \le b) \end{aligned}$$

It is straightforward to check that every $x(t)$ of any of these forms is a solution of the IVP.

**3. (a)** Using Lipschitz continuity with respect to $x$, we have

$$\begin{aligned} U(t) = |x_1(t) - x_2(t)| &= \left|\left(a + \int_{t_0}^t f(s, x_1(s))\,ds\right) - \left(a + \int_{t_0}^t f(s, x_2(s))\,ds\right)\right| \\ &= \left|\int_{t_0}^t [f(x, x_1(s)) - f(x, x_2(s))]\,ds\right| \le \int_{t_0}^t |f(x, x_1(s)) - f(x, x_2(s))|\,ds \\ &\le L\int_{t_0}^t |x_1(s) - x_2(s)|\,ds = L\int_{t_0}^t U(s)\,ds. \end{aligned}$$

**(b)** From the Fundamental Theorem of Calculus and (a), we have $V'(t) = U(t) \le V(t)$. Also,

$$V(t_0) = \varepsilon + L\int_{t_0}^{t_0} U(s)\,ds = \varepsilon.$$

**(c)** Dividing by $V(t)$ and integrating gives

$$\ln\left(\frac{V(T)}{\varepsilon}\right) = \ln\left(\frac{V(T)}{V(t_0)}\right) = \int_{t_0}^T \frac{V'(t)}{V(t)}\,dt \le \int_{t_0}^T L\,dt = L(T - t_0),$$

so $V(T) \le \varepsilon\exp[L(T - t_0)]$.

**(d)** For every $\varepsilon > 0$, we have $U(T) \leq V(T) \leq \varepsilon \exp[L(T - t_0)] \to 0$ as $\varepsilon$ goes to 0. Since $U(T)$ is nonnegative by definition, this means $U(T) = 0$, so $x_1(T) = x_2(T)$. Since $T \in [t_0, t_0 + b]$ was arbitrary, $x_1 = x_2$ on $[t_0, t_0 + b]$.

**4.** Let $(x_-, x_+)$ and $(y_-, y_+)$ be the domains of $x$ and $y$, respectively. Since $h \circ x$ is continuous, the IVP

$$\begin{cases} j' = (h \circ x)(j) \\ j(0) = 0 \end{cases}$$

has a solution $j$, and $j$ is increasing because $h > 0$. Define $Y := x \circ j$, and note that $Y(0) = x(j(0)) = x(0) = a$, and

$$\dot{Y}(t) = j'(t)\dot{x}(j(t)) = h(x(j(t))f(x(j(t))) = g(x(j(t))) = g(Y(t)),$$

so $Y$ satisfies the same IVP as $y$. By uniqueness, this means that $Y = y$ on their common domain of definition, so $y(t) = x(j(t))$ for $t \in \operatorname{dom}(j) =: (j_-, j_+)$, $j$'s maximal interval of existence. By definition, the range of $j$ is contained in the $(x_-, x_+)$, and by the maximality of $y$'s domain, we know that $(j_-, j_+) \subseteq (y_-, y_+)$. We need to show that, in fact, $(j_-, j_+) = (y_-, y_+)$; by a time reversal argument, it suffices to show that $j_+ = y_+$.

If $j_+ = \infty$, then we're done, so suppose that $j_+ < \infty$. Then the results of this section applied to the $j$-IVP imply that $j(t) \uparrow x_+$ as $t \uparrow j_+$. Either $x([0, x_+))$ is contained in a compact subset of $\Omega$ or it isn't. Suppose the first case occurs. Then the continuity of $h$ on $\Omega$ implies that $(h \circ x)([0, x_+))$ is bounded, so $j'([0, j_+))$ is bounded. Since $j_+ < \infty$, this means that $x_+ < \infty$. This contradicts the results of this section applied to the $x$-IVP. This puts us in the second case. Since $j(t) \uparrow x_+$ as $t \uparrow j_+$, we have $y([0, j_+)) = x(j([0, j_+))) = x([0, x_+))$, which means that the continuous function $y$ can't be defined at $j_+$. Hence, $j_+ = y_+$.

**5. (a)** Suppose there exists $t \geq t_0$ such that $x(t) \geq y(t)$. By the continuity of $x$ and $y$ there must be a first such time $t^*$. Since $x(t_0) = a < b = y(t_0)$, we know $t^* > t_0$ and $x(t^*) = y(t^*)$. Note that $\dot{x}(t^*) = f(t, x(t^*)) = f(t, y(t^*)) < g(t, y(t^*)) = \dot{y}(t^*)$, so $x(t) > y(t)$ for $t$ just smaller than $t^*$. This contradicts the definition of $t^*$, and this contradiction implies the desired result.

**(b)** Given $\varepsilon > 0$, set $\tilde{b} = b + \varepsilon$ and $\tilde{g}(t, p) = g(t, p) + \varepsilon$, and let $\tilde{y}$ be the solution of the IVP

$$\begin{cases} \dot{\tilde{y}} = \tilde{g}(t, \tilde{y}) \\ \tilde{y}(t_0) = \tilde{b}. \end{cases}$$

Since $f(t, p) \leq g(t, p) < \tilde{g}(t, p)$ and $a \leq b < \tilde{b}$, the results of (a) imply that $x(t) < \tilde{y}(t)$ for every $t \geq t_0$. By the Theorem on Continuous Dependence, $\tilde{y}(t) \to y(t)$ as $\varepsilon \downarrow 0$, so $x(t) \leq y(t)$ for every $t \geq t_0$.

(Note that Exercise 2 provides a counterexample to any purported proof that fails to utilize Lipschitz continuity.)

**6.** Fix $t \in \mathbb{R}$, and consider the function $g : \mathbb{R}^n \to \mathbb{R}^n$ defined by $g(p) = f(t, p)$. Let $p, q \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$ be given. Since $f$ is continuous, every IVP associated with the ODE has a solution, so we can let $x$ be a solution of the ODE satisfying $x(0) = p$ and let $y$ be a solution of the ODE satisfying $y(0) = q$. By hypothesis, $\alpha x + \beta y$ satisfies the ODE, so

$$\begin{aligned} g(\alpha p + \beta q) &= f(t, \alpha p + \beta q) = f(t, \alpha x(0) + \beta y(0)) = f(t, (\alpha x + \beta y)(0)) \\ &= (\alpha x + \beta y)'(0) = \alpha \dot{x}(0) + \beta \dot{y}(0) = \alpha f(t, x(0)) + \beta f(t, y(0)) \\ &= \alpha f(t, p) + \beta f(t, q) = \alpha g(p) + \beta g(q). \end{aligned}$$

This shows that $g$ is linear, so $g \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$. Set $A(t) = g$. This defines a function $A : \mathbb{R} \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ satisfying $A(t)p = g(p) = f(t, p)$.

**7.** A suitable collection is:

$$\left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}; \begin{bmatrix} a & 0 & 0 & 0 \\ 1 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{bmatrix}; \begin{bmatrix} a & 0 & 0 & 0 \\ 1 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 1 & b \end{bmatrix}; \begin{bmatrix} a & 0 & 0 & 0 \\ 1 & a & 0 & 0 \\ 0 & 1 & a & 0 \\ 0 & 0 & 0 & b \end{bmatrix}; \begin{bmatrix} a & 0 & 0 & 0 \\ 1 & a & 0 & 0 \\ 0 & 1 & a & 0 \\ 0 & 0 & 1 & a \end{bmatrix}; \right.$$

$$\begin{bmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}, (b \neq 0); \begin{bmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 1 & c \end{bmatrix}, (b \neq 0); \begin{bmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & c & -d \\ 0 & 0 & d & c \end{bmatrix}, (b, d \neq 0);$$

$$\left. \begin{bmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 1 & 0 & a & -b \\ 0 & 1 & b & a \end{bmatrix}, (b \neq 0) \right\}$$

**8. (a)** The series formula gives

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+ \frac{t^3}{6} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{t^4}{24} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ t & 1 - t^2/2 & t^2/2 & t \\ t & -t^2/2 & 1 + t^2/2 & t \\ 0 & -t & t & 1 \end{bmatrix}.$$

**(b)** Since $A^2 = 10A$, $A^k = 10^{k-1}A$ for all $k \geq 1$, so

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = I + \sum_{k=1}^{\infty} \frac{t^k 10^{k-1} A}{k!} = I + \frac{A}{10} \sum_{k=1}^{\infty} \frac{t^k 10^k}{k!} = I + \frac{A}{10}(e^{10t} - 1)$$

$$= \frac{1}{10} \begin{bmatrix} e^{10t} + 9 & e^{10t} - 1 & e^{10t} - 1 & e^{10t} - 1 \\ 2e^{10t} - 2 & 2e^{10t} + 8 & 2e^{10t} - 2 & 2e^{10t} - 2 \\ 3e^{10t} - 3 & 3e^{10t} - 3 & 3e^{10t} + 7 & 3e^{10t} - 3 \\ 4e^{10t} - 4 & 4e^{10t} - 4 & 4e^{10t} - 4 & 4e^{10t} + 6 \end{bmatrix}.$$

**9. (a)** Since $e^{tA}$ and $e^{tB}$ are contractions, the results of this section indicate that there are constants $k_1, b_1, k_2, b_2 > 0$ such that $\|e^{tA}x\| \leq k_1 e^{-tb_1}\|x\|$ and $\|e^{tB}x\| \leq k_2 e^{-tb_2}\|x\|$ for all $x \in \mathbb{R}^n$ and $t \geq 0$. If $A$ and $B$ commute, then so do $tA$ and $tB$, so the lemma in Section 2.1 indicates that $e^{t(A+B)} = e^{tA}e^{tB}$. Thus,

$$\|e^{t(A+b)}x\| = \|e^{tA}e^{tB}x\| \leq k_1 e^{-tb_1}\|e^{tB}x\| \leq k_1 e^{-tb_1} k_2 e^{-tb_2}\|x\| = ke^{-tb}\|x\|,$$

where $k = k_1 k_2 > 0$ and $b = b_1 + b_2 > 0$. Thus, $e^{t(A+b)}$ is a contraction.

**(b)** Let

$$A = \begin{bmatrix} -1 & 0 \\ 3 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 3 \\ 0 & -1 \end{bmatrix}.$$

The only eigenvalue of $A$ or $B$ is $-1$, so $e^{tA}$ and $e^{tB}$ are contractions. On the other hand,

$$A + B = \begin{bmatrix} -2 & 3 \\ 3 & -2 \end{bmatrix},$$

which has 1 as one of its eigenvalues, so $e^{t(A+B)}$ is not a contraction.

**10.** The listed alternatives are insufficient. To determine the correct list of alternatives, we view solutions with respect to the basis corresponding to real canonical form. We first determine all possible asymptotic behaviors of solutions confined to $\mathcal{E}^u$, $\mathcal{E}^s$, or $\mathcal{E}^c$.

If $x(t)$ is a solution in $\mathcal{E}^u$, then its components are all of the form $p(t)e^{at}(\alpha \cos bt + \beta \sin bt)$ where $p$ is a polynomial and $a > 0$. Furthermore, if $b \neq 0$ (and $\alpha^2+\beta^2 \neq 0$ and $p(t)$ is not identically zero) then there is a complementary component of the form $\pm p(t)e^{at}(\beta \cos bt - \alpha \sin bt)$ with the same $p, a, b, \alpha$, and $\beta$. This tells us that either $x(t)$ is identically zero or $x(t)$ approaches 0 as $t \downarrow -\infty$ and diverges to $\infty$ as $t \uparrow \infty$. Similarly, any solution in $\mathcal{E}^s$ is either identically zero or converges to zero in forward time and diverges to $\infty$ in backward time.

The analysis of possible behaviors for a solution $x(t)$ in $\mathcal{E}^c$ is slightly more complicated. Each component of $x(t)$ is of the form $p(t)(\alpha \cos bt + \beta \sin bt)$ with $p$ a polynomial, and for each such component with $b \neq 0$ (and $\alpha^2 + \beta^2 \neq 0$ and $p(t)$ not identically zero) there is a complementary component of the form $\pm p(t)(\beta \cos bt - \alpha \sin bt)$ with the same $p, b, \alpha$, and $\beta$. One possibility is, of course, that all components are identically zero, so $x(t)$ is identically zero. If each $p(t)$ is a constant but there is some component that is not identically zero, then $x(t)$ is bounded and bounded away from zero. If there is a nontrivial component whose polynomial is nonconstant, then $x(t)$ diverges to infinity in both forward and backward time.

We can now determine the behavior of arbitrary solutions by using the fact that each solution $x(t)$ is a sum of solutions $x_u(t)$ in $\mathcal{E}^u$, $x_s(t)$ in $\mathcal{E}^s$, and $x_c(t)$ in $\mathcal{E}^c$. By considering all possible combinations of the behaviors noted above (and using the fact that $x_u(t)$, $x_s(t)$, and $x_c(t)$ lie in different subspaces and therefore cannot cancel one another out), we see that Hirsch and Smale's list will be correct if and only if it is supplemented with the following alternatives:

**(d)** $x(t) = 0$ for every $t \in \mathbb{R}$;

**(e)** $\lim_{t\downarrow-\infty} |x(t)| = \lim_{t\uparrow\infty} |x(t)| = \infty$;

**(f)** $\lim_{t\downarrow-\infty} |x(t)| = \infty$ and there exist constants $M, N > 0$ such that $M < |x(t)| < N$ for all $t > 0$;

**(g)** $\lim_{t\uparrow\infty} |x(t)| = \infty$ and there exist constants $M, N > 0$ such that $M < |x(t)| < N$ for all $t < 0$.

**11.** The trace of $A(t)$ is $-1/2$ and the determinant is $1/2$, so the eigenvalues are $(-1 \pm i\sqrt{7})/4$, which each have negative real part. Furthermore,

$$\begin{aligned} A(t)\begin{bmatrix} -\cos t \\ \sin t \end{bmatrix} e^{t/2} &= \left( \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} \cos^2 t & -\sin t \cos t \\ -\sin t \cos t & \sin^2 t \end{bmatrix} \right) \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix} e^{t/2} \\ &= \left( \begin{bmatrix} \cos t + \sin t \\ \cos t - \sin t \end{bmatrix} + \frac{3}{2}\begin{bmatrix} -\cos t \\ \sin t \end{bmatrix} \right) e^{t/2} \\ &= \left( \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -\cos t \\ \sin t \end{bmatrix} \right) e^{t/2} = \frac{d}{dt}\begin{bmatrix} -\cos t \\ \sin t \end{bmatrix} e^{t/2}, \end{aligned}$$

so

$$\begin{bmatrix} -\cos t \\ \sin t \end{bmatrix} e^{t/2}$$

is a solution that becomes unbounded in forward time.

**12.** By the results of this section, the equation $\dot{x} = A(t)x$ has Floquet multipliers whose product is

$$\exp\left(\int_0^{2\pi} (2 - \cos t + \sin t)\, dt\right) = e^{4\pi}.$$

Since this product is larger than 1, one of the Floquet multipliers must be larger than 1 in absolute value. A theorem from this section says that this means that there is a nontrivial solution of the ODE that grows in absolute value by a factor greater than 1 each $2\pi$ units of time. This solution becomes unbounded as $t$ goes to $\infty$.

**13.**

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a | F | F | F | F | F | T | T | T | T | T | T | T | T | T | T | T | T | T |
| b | F | F | F | F | F | F | F | F | F | T | T | T | T | F | F | F | T | T |
| c | F | F | T | T | T | F | F | T | T | F | F | T | T | T | T | T | T | T |
| d | F | F | F | F | F | F | F | F | F | F | F | F | F | T | T | T | T | T |
| e | F | T | F | T | T | F | T | F | T | F | T | F | T | F | T | T | F | T |
| f | F | T | F | F | T | F | T | F | T | F | F | F | T | F | F | T | F | T |

**14.** Let $f$, $g$, and $V$ be as in the hint, and consider the ODE $\dot{x} = f(t, x)$. Note that

$$\int_0^t [g(\tau)]^2\, d\tau < \int_0^\infty [g(\tau)]^2\, d\tau \le \int_0^\infty e^{-2\tau}\, d\tau + 2\sum_{n=1}^{\infty} 2^{-n} = \frac{1}{2} + 2 = \frac{5}{2},$$

so

$$V(t, x) = \frac{x^2}{[g(t)]^2}\left[3 - \int_0^t [g(\tau)]^2\, d\tau\right] \ge \frac{x^2}{1}\left[3 - \frac{5}{2}\right] = \frac{x^2}{2}.$$

This shows that $V(t, x)$ is positive definite. Also,

$$\begin{aligned} \dot{V}(t, x) &= \frac{2xf(t,x)[g(t)]^2 - 2x^2 g(t)g'(t)}{[g(t)]^4}\left[3 - \int_0^t [g(\tau)]^2\, d\tau\right] - x^2 \\ &= \frac{2x2g(t)g'(t) - 2x^2 g(t)g'(t)}{[g(t)]^4}\left[3 - \int_0^t [g(\tau)]^2\, d\tau\right] - x^2 = -x^2, \end{aligned}$$

so $\dot{V}(t, x)$ is negative definite.

The ODE

$$\dot{x} = f(t, x) = \frac{g'(t)}{g(t)} x$$

is separable and its general solution is $x(t) = cg(t)$. Thus, no solution of the equation except for the zero solution converges to 0 in forward time. Hence, 0 is not asymptotically stable.

**15.** Let $V = x^2 + y^2$. Note that $V$ is positive definite and $\dot{V} = 2x\dot{x} + 2y\dot{y} = 2x(-x^3 + 2y^3) + 2y(-2xy^2) = -2x^4$, which is negative semidefinite. This shows that $(0, 0)$ is Lyapunov stable.

Now let $\mathcal{D}$ be the closed disc of radius 1 centered at the origin. The set where $\dot{V} = 0$ lies on the $y$-axis, and $\dot{x}$ is nonzero everywhere on that set except for the origin, so the union of the set of complete orbits along which $\dot{V} = 0$ is $\{(0,0)\}$. By LaSalle's Invariance Principle, the $\omega$-limit set of each point in $\mathcal{D}$ is therefore contained in $\{(0,0)\}$. We claim that, in fact, every solution starting in $\mathcal{D}$ converges to 0 in forward time. If not, then there would be a sequence of arbitrarily late points on the orbit bounded away from the origin, and some subsequence of this sequence would converge to an $\omega$-limit point other than the origin. This can't happen, so the claim holds, and the origin is asymptotically stable.

**16. (a)** Define $h : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$h\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y - 4x^2/7 \end{bmatrix}.$$

It is easy to check that

$$h^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y + 4x^2/7 \end{bmatrix},$$

so $h$ is a homeomorphism. Furthermore,

$$\begin{aligned} h^{-1}\left(F\left(h\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right)\right) &= h^{-1}\left(F\left(\begin{bmatrix} x \\ y - 4x^2/7 \end{bmatrix}\right)\right) = h^{-1}\left(\begin{bmatrix} -x/2 \\ 2y - x^2/7 \end{bmatrix}\right) \\ &= \begin{bmatrix} -x/2 \\ 2y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}, \end{aligned}$$

so $h$ is a topological conjugacy between $F$ and $A$.

**(b)** Define $h : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$h\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y - x^2/3 \end{bmatrix},$$

which is a homeomorphism with inverse

$$h^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y + x^2/3 \end{bmatrix}.$$

Since $h$ is differentiable, we can check whether it provides a conjugacy between the given flows by checking that $F(h(u)) = Dh(u)Au$. Calculating,

$$F\left(h\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right) = F\left(\begin{bmatrix} x \\ y - x^2/3 \end{bmatrix}\right) = \begin{bmatrix} -x/2 \\ 2y + x^2/3 \end{bmatrix},$$

and

$$Dh\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2x/3 & 1 \end{bmatrix} \cdot \begin{bmatrix} -x/2 \\ 2y \end{bmatrix} = \begin{bmatrix} -x/2 \\ 2y + x^2/3 \end{bmatrix},$$

so $h$ is a conjugacy and the flows are conjugate.