Newton's Binomial Formula

The choose function. Suppose we have a coupon for a large pizza with (exactly) three toppings and the pizzeria offers 10 choices of toppings. We want to know how many options we have, i.e. how many pizzas are there with exactly three toppings?

We write down the list of toppings as a set:

{ pepperoni, onions, olives, mushrooms, . . . , jalapenos }

We have 10 choices for the first topping, then 9 for the second topping and 8 for the third topping. All in all $10 \cdot 9 \cdot 8 = 720$ choices. But notice that if we say "onions, mushrooms, jalapenos" or "mushroom, onions, jalapenos" we get the same pizza. So if we want to count the different pizzas (not the different ways to say three toppings) we must divide by the number of options to say the same three toppings in a different orders. We can represent the toppings as $1, 2, 3$ and count the number of ways to order them: $\{1, 2, 3\}, \{1, 3, 2\}, \{2, 1, 3\}, \{2, 3, 1\}, \{3, 1, 2\}, \{3, 2, 1\}$ all in all 6 different ways. Here's a way to count it: There are three options for the first number, two options for the second and only one number left for the third so $3 \cdot 2 \cdot 1 = 6$ ways to order three toppings. Therefore the number of different pizzas we can order is $\frac{720}{6} = 120$.

We can translate the pizza problem to the following abstract problem: Consider the set $\{1, \dots, 10\}$, how many three element subsets does it have? Notice that when we say 'subset' we automatically disregard the order of the elements in the set. In other words we want the number of 'ways' to choose three elements out of ten. This number is denoted $\binom{10}{3}$ and read 10 choose 3. We've calculated that $\binom{10}{3} = 120$.

In general, $\binom{n}{k}$ for natural numbers $n \geq k$ denotes the number of k-element subsets of the set $\{1, \dots, n\}$ (the number of k topping pizzas where we have n choices for toppings). Notice that $\binom{n}{0} = 1$ - indeed, if our coupon is for a plain pizza it doesn't matter how many toppings there are - we only have one choice. And $\binom{n}{n} = 1$ (what's the pizza scenario here?).

To compute a formula for $\binom{n}{k}$ in general we proceed exactly as in the $\binom{10}{3}$ case. We first choose a k element subset: n choices for the first element, $n - 1$ choices for the second element, $n - 2$ choices for the third element, . . . , $n - k + 1$ choices for the k-th element. So far we have $n(n-1)\cdots(n-k+1)$ subsets. But now we must remember to divide by the number of ways to order the elements. As before there are k options to chose from for the first element, $k-1$ options for the second element, $\ldots$, 2 options for the $k-1$ element, and 1 option for the last element. So there are $k \cdot (k-1) \cdots 2 \cdot 1$ sequences containing the same numbers but arranged in different orders. Therefore the number of subsets of $\{1, \ldots n\}$ which contain k elements is $\frac{n(n-1)\ldots(n-k)}{k(k-1)\cdots 2\cdot 1}$.

Recall the factorial function: $0! = 1$, $1! = 1$, $2! = 2 \cdot 1$, $3! = 3 \cdot 2 \cdot 1$ and in general $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$. We can use this function to rewrite the formula for the choose function as :

$$\boxed{\binom{n}{k} = \frac{n!}{k!(n-k)!}}$$

Remark 0.1. *From this formula it's easy to see that* $\binom{n}{k} = \binom{n}{n-k}$ *(Just plug in* $n-k$ *for* k*). Can you see it in a different way?*

The binomial formula. We've all seen the following rule in high-school:

$$\boxed{(a+b)^2 = a^2 + 2ab + b^2}$$

How can we simplify: $(a+b)^3$?

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)(a+b) \\ &= a(a+b)(a+b) + b(a+b)(a+b) \\ &= a[a(a+b) + b(a+b)] + b[a(a+b) + b(a+b)] \\ &= a[aa + ab + ba + bb] + b[aa + ab + ba + bb] \\ &= aaa + aab + aba + abb + baa + bab + bba + bbb\end{aligned}$$

We get a sum of products of as and bs. In each summand the number of as and bs multiplied together is 3. If we change the order of multiplication so that the as always appear before the bs a general element in the sum is $a^i b^j$ where $i + j = 3$. Another way to write this is $a^i b^{3-i}$.

$$\begin{aligned}(a+b)^3 &= a^3b^0 + a^2b^1 + a^2b^1 + a^1b^2 + a^2b^1 + a^1b^2 + a^1b^2 + a^0b^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

In a summand each a or b comes from one of the parenthesis. We get aab by multiplying the a in the first parenthesis, with an a in the second parenthesis with a b in the third parenthesis. Another way to get $a^2b = aba$ is to multiply the a in the first parenthesis with the b in the second parenthesis with the a in the third parenthesis. In conclusion, we see a^2b in the expansion when two of the parenthesis contribute an a and one of them contributes a b. Therefore, the number of times we see a^2b in the expansion equals the number of ways we can chose 2 of the three parenthesis: we're choosing the parenthesis which contribute the a - the rest contribute a b. For examplea^2b appears three times since there are three ways to chose two out of three parenthesis: first and second, first and third, and second and third. Indeed $\binom{3}{2} = \frac{3!}{2!(3-2)!} = \frac{6}{2} = 3$.

$$\begin{aligned}(a+b)^3 &= \tbinom{3}{3}a^3b^0 + \tbinom{3}{2}a^2b + \tbinom{3}{1}a^1b^2 + \tbinom{3}{0}a^0b^3 \\ &= 1 \cdot a^3b^0 + 3 \cdot a^2b^1 + 3 \cdot a^1b^2 + 1 \cdot a^0b^3\end{aligned}$$

Now we are ready to expand $(a+b)^n$ for a general natural number n:

$$\begin{aligned}(a+b)^n &= \underbrace{(a+b)\cdot(a+b)\cdots(a+b)}_{n \text{ times}} \\ &= \tbinom{n}{n}a^nb^0 + \tbinom{n}{n-1}a^{n-1}b^1 + \tbinom{n}{n-2}a^{n-2}b^2 + \cdots + \tbinom{n}{2}a^2b^{n-2} + \tbinom{n}{1}a^1b^{n-1} + \tbinom{n}{0}a^0b^n\end{aligned}$$

Using the sigma notation we can write this in a more compact way:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{n-i} a^{n-i}b^i$$

We can change the order of summation (going from b^n to a^n) to get the formula:

$$\boxed{(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}}$$