A MONOTONE FUNCTION IS INTEGRABLE

Theorem. *Let f be a monotone function on $[a,b]$ then f is integrable on $[a,b]$.*

Proof. We will prove it for monotonically decreasing functions. The proof for increasing functions is similar.
First note that if f is monotonically decreasing then $f(b) \leq f(x) \leq f(a)$ for all $x \in [a,b]$ so f is bounded on $[a,b]$.
Denote by P_n the partition of $[a,b]$ into n equal intervals.

$$P_n = \{a < a + \frac{b-a}{n} < a + 2\frac{b-a}{n} < \cdots < x_k = a + k\frac{b-a}{n} < \cdots < a + n\frac{b-a}{n} = b\}$$

We compute:

$$x_k - x_{k-1} = a + k\frac{b-a}{n} - \left(a + (k-1)\frac{b-a}{n}\right) = \frac{b-a}{n}$$

Since f is monotonic:

$$m_k = \inf\{f(x) | x_{k-1} \leq f(x) \leq x_k\} = f(x_k)$$

$$M_k = \sup\{f(x) | x_{k-1} \leq f(x) \leq x_k\} = f(x_{k-1})$$

Therefore,

$$L(f, P_n) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = \sum_{k=1}^{n} f(x_k)\frac{b-a}{n}$$
$$U(f, P_n) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = \sum_{k=1}^{n} f(x_{k-1})\frac{b-a}{n}$$

Thus,

$$U(f,P_n) - L(f,P_n) =$$
$$\sum_{k=1}^{n} f(x_{k-1})\frac{b-a}{n} - \sum_{k=1}^{n} f(x_k)\frac{b-a}{n} =$$
$$\frac{b-a}{n}\left(\sum_{k=1}^{n} f(x_{k-1}) - \sum_{k=1}^{n} f(x_k)\right) =$$
$$\frac{b-a}{n}\left(\sum_{k=0}^{n-1} f(x_k) - \sum_{k=1}^{n} f(x_k)\right) =$$
$$\frac{b-a}{n}(f(x_0) - f(x_n)) = \frac{b-a}{n}(f(b) - f(a))$$

Therefore

$$\lim_{n\to\infty} U(f, P_n) - L(f, P_n) = \lim_{n\to\infty} \frac{(b-a)(f(a)-f(b))}{n} = 0$$

By the sequential characterization of integrability, f is integrable on $[a, b]$. $\square$