ROLLE'S MEAN VALUE THEOREM

Theorem (Rolle). *Let f be a continuous function on $[a, b]$ which is differentiable on (a, b). Suppose $f(a) = f(b)$. Then there exists a point $c \in (a, b)$ such that $f'(c) = 0$*

Proof. Since f is continuous on the closed interval $[a, b]$, by Weierstrauss' extreme value theorem, f assumes a minimum m and a maximum M on this interval. That is, there are points $c, d \in [a, b]$ such that:

i $f(x) \leq M$ for all $x \in [a, b]$, and $f(c) = M$.
ii $f(x) \geq m$ for all $x \in [a, b]$, and $f(d) = m$.

If $m = M$ then $f(x) = m$ for all $x \in [a, b]$ and we've computed that $f'(x) = 0$ for all $x \in (a, b)$ so in this case we are done.
Now assume that $m < M$. $f(a) \neq m$ or $f(a) \neq M$ assume without loss of generality that $f(a) \neq M$. Since $f(b) = f(a)$ then $f(b) \neq M$. Therefore $c \neq a$ and $c \neq b$ hence $a < c < b$ and so f is differentiable at c. c is the source of the maximum so it is a local maximum (a maximum which is not at an endpoint). We appeal to Fermat's theorem which says:
If f is defined on $(c - t_0, c + t_0)$ for some t_0, f differentiable at c, and f has a local maximum or minimum at c then $f'(c) = 0$
Appealing to this theorem, we have: $f'(c) = 0$. The case where $f(a) \neq m$ is similar. In this case it follows that $f'(d) = 0$. □