MATH 3210 - SUMMER 2008 - PRACTICE MIDTERM

You have an hour and a half to complete this test. The maximum grade is 100.

question	**grade**	**out of**
1		33
2		33
3a		16
3b		8
3c		6
3d		4
total		100

Student Number: ____________________________

(1) (33 pts) Using the definition of a convergent sequence prove the following theorem (Do not appeal to any theorem):

If $\{a_n\}_{n=1}^{\infty}$ converges to a and $a > 0$ then there is an $N_1 \in \mathbb{N}$ such that for all $n > N_1$: $a_n > 0$.

Proof.

NTS: There exists a natural number N_1 such that for all $n > N_1 : a_n > 0$

We know:

I. for all $\varepsilon > 0$ there is an $N(\varepsilon) \in \mathbb{N}$ such that for all $n > N(\varepsilon)$: $|a_n - a| < \varepsilon$

II. $a > 0$

Let $\varepsilon = \frac{a}{2}$ by II. this is a positive number.

By I. there is a natural number $N_1 = N(\frac{a}{2})$ such that for all $n > N_1$:

$$|a_n - a| < \tfrac{a}{2}$$
$$-\tfrac{a}{2} < a_n - a < \tfrac{a}{2}$$
$$a - \tfrac{a}{2} < a_n < a + \tfrac{a}{2}$$
$$\tfrac{a}{2} < a_n < \tfrac{3a}{2}$$
$$\tfrac{a}{2} < a_n$$

Thus $a_n > \frac{a}{2} > 0$ □

(2) (33 pts) Consider the following sequence defined inductively:

$$a_1 = 1$$
$$a_{n+1} = \sqrt[3]{(a_n)^2 + 2a_n}$$

Prove that $\{a_n\}_{n=1}^{\infty}$ converges and find its limit.

Proof. We will show that $\{a_n\}_{n=1}^{\infty}$ is monotonically increasing and bounded above. By the Monotone Convergence Theorem it converges to some limit which we denote by L.

If $\lim_{n\to\infty} a_n = L$ then for every subsequence a_{n_k} we have $\lim_{k\to\infty} a_{n_k} = L$

In particular $\lim_{k\to\infty} a_{k+1} = L$. On the other hand, by the main limit theorem $\lim_{n\to\infty} \sqrt[3]{(a_n)^2 + 2a_n} = \sqrt[3]{L^2 + 2L}$. Since the limit is unique:

$$L = \sqrt[3]{L^2 + 2L}$$
$$L^3 = L^2 + 2L$$
$$L^3 - L^2 - 2L = 0$$
$$L(L^2 - L - 2) = 0$$

The other two roots of this equation are $\frac{1 \pm \sqrt{1+8}}{2} = 2, -1$. Since we show that a_n is monotonically increasing then for all n: $a_n \geq a_1 = 1$ thus $L \neq 0, -1$ hence $L = 2$. So once we prove that a_n converges its limit must be 2.

Claim. *For all $n \in \mathbb{N}$: $a_{n+1} \geq a_n$*

Proof of Claim. We prove this by induction.

- Basis: We check this for $n = 1$: $a_2 \geq^? a_1$

  $a_1 = 1$, $a_2 = \sqrt[3]{3}$ and $3 > 1$ implies $\sqrt[3]{3} > \sqrt[3]{1} = 1$
- Induction Hypothesis: $a_{n+1} \geq a_n$

- Induction Step: $a_{n+2} \geq^? a_{n+1}$

$$
\begin{aligned}
&a_{n+2} \geq a_{n+1} \Leftrightarrow \\
&\sqrt[3]{(a_{n+1})^2 + 2a_{n+1}} \geq \sqrt[3]{(a_n)^2 + 2a} \Leftrightarrow \\
&(a_{n+1})^2 + 2a_{n+1} \geq (a_n)^2 + 2a_n \Leftrightarrow \\
&a_{n+1}(a_{n+1} + 2) \geq a_n(a_n + 2)
\end{aligned}
$$

And since $a_{n+1} > a_n$, $a_{n+1} + 2 > a_n + 2$ the last line is a true statement.

□

Claim. $a_n < 2$ *for all* $n \in \mathbb{N}$

Proof of claim. We prove this by induction.

- Basis: We check this for $n = 1$: $a_1 = 1 < 2$
- Induction Hypothesis: $a_n < 2$
- Induction Step: $a_{n+1} <^? 2$

$$
\begin{aligned}
&a_{n+1} < 2 \Leftrightarrow \\
&\sqrt[3]{(a_n)^2 + 2a_n} < 2 \Leftrightarrow \\
&(a_n)^2 + 2a_n < 8
\end{aligned}
$$

And since $a_n < 2$ then $(a_n)^2 + 2a_n < 2^2 + 2 \cdot 2 = 8$

□

We proved a_n converges (by the monotone convergence theorem) and its limit is 2 □

(3) (34 pts) For each of the following statements, determine if they are true or false. If they are true, prove them. You are allowed and encouraged to appeal to the theorems proven in class (without proof) as long as you quote them in full. If the statement is false find a counter example.

(a) (16 pts) True/False:

If $\lim_{n\to\infty} a_n = 0$ and $\{b_n\}_{n=1}^{\infty}$ satisfies:

$$-\frac{1}{n^2} \leq b_n \leq a_n + \frac{3}{n}$$

for all $n \in \mathbb{N}$. Then $\{b_n\}_{n=1}^{\infty}$ converges.

The statement is true.

Proof. Since $\{a_n\}_{n=1}^{\infty}$ converges to 0 and $\frac{3}{n}$ converges to 0 then by the main limit theorem

$$\lim_{n\to\infty} a_n + \frac{3}{n} = \lim_{n\to\infty} a_n + \lim_{n\to\infty} \frac{3}{n} = 0 + 0 = 0$$

Furthermore, $\lim_{n\to\infty} -\frac{1}{n^2} = 0$ (don't need a proof here).

By the sandwich theorem if $a_n \leq b_n \leq c_n$ for all n and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then b_n also converges to L.

Therefore, taking $a_n = -\frac{1}{n^2}$, $c_n = a_n + \frac{3}{n}$ we get that b_n converges to 0. □

(b) (8 pts) True/False:

Suppose $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ are sequences which satisfy the following properties:

(i) $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = 0$

(ii) $b_n \neq 0$ for all $n \in \mathbb{N}$

then $\lim_{n\to\infty} (a_n)^{b_n} = 0$

False.

Consider the sequences $a_n = b_n = \frac{1}{n}$.

We showed in class that $\lim_{n\to\infty} \dfrac{1}{n} = 0$. And, $b_n \neq 0$ for all $n \in \mathbb{N}$. Therefore, these sequences satisfy all of the assumptions. However, we claim that they don't satisfy the conclusion:

$$(a_n)^{b_n} = \left(\frac{1}{n}\right)^{\frac{1}{n}} = \sqrt[n]{\frac{1}{n}} = \frac{1}{\sqrt[n]{n}}$$

We've shown in the homework that $\lim_{n\to\infty} \sqrt[n]{n} = 1$. By the main limit theorem

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} \sqrt[n]{n}} = \frac{1}{1} = 1$$

Therefore this statement is false.

(c) (6 pts) True/False:

If $\{a_n\}_{n=1}^{\infty}$ diverges then $\{a_n\}_{n=1}^{\infty}$ is not bounded.

False.

Consider the sequence $a_n = (-1)^n = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$

We proved in class that this sequence diverges.

Furthermore, for all n: $|a_n| = |(-1)^n| = 1$ so a_n is bounded.

(d) (4 pts) True/False:

There is a strictly monotonically increasing sequence $\{a_n\}_{n=1}^{\infty}$ of natural numbers such that the sequence $b_n = \cos(a_n)$ converges.

True.

Proof. Consider the sequence $c_k = \cos(k)$.

$|c_k| = |\cos(k)| \leq 1$ therefore it is bounded.

We appeal to the Bolzano-Weierstrauss theorem:

Any bounded sequence $\{c_k\}_{k=1}^{\infty}$ has a convergent subsequence: $\{c_{k_n}\}_{n=1}^{\infty}$

Therefore there is some strictly monotonic sequence of natural numbers k_n such that $c_{k_n} = \cos(k_n)$ converges. Thus set: $a_n = k_n$ and it satisfies what we want. $\square$