MATH 3210 - SUMMER 2008 - ASSIGNMENT #10

Solution

Exercise. *Prove that for* $0 < x < y < \frac{\pi}{2}$*:* $\frac{1}{\cos^2(x)} < \frac{\tan(y)-\tan(x)}{y-x} < \frac{1}{\cos^2(y)}$. *(hint: start by proving that in this domain the derivative of* tan *is monotonic)*

Proof. $f(x) = \tan(x)$ is differentiable in $(0, \frac{\pi}{2})$. For $0 < x < y < \frac{\pi}{2}$, f is differentiable on $[x, y]$. By the mean value theorem there is a point $c \in (x, y)$ such that

$$\frac{\tan(y) - \tan(x)}{y - x} = \tan'(c)$$

We proved in class that $\tan'(c) = \frac{1}{\cos^2(c)}$ Now:

$$\begin{aligned}
&x < c < y \Rightarrow \\
&(\star) \qquad \cos(x) > \cos(c) > \cos(y) \Rightarrow \\
&(\star\star) \qquad (\cos(x))^2 > (\cos(c))^2 > (\cos(y))^2 \implies \\
&\tfrac{1}{(\cos(x))^2} < \tfrac{1}{(\cos(c))^2} < \tfrac{1}{(\cos(y))^2} \\
&\tfrac{1}{(\cos(x))^2} < \tan'(c) < \tfrac{1}{(\cos(y))^2} \\
&\tfrac{1}{(\cos(x))^2} < \tfrac{\tan(y)-\tan(x)}{y-x} < \tfrac{1}{(\cos(y))^2}
\end{aligned}$$

$\star$ - since cos is monotonically decreasing in this interval

$\star\star$ - since all the numbers involved are positive

□