MATH 3210 - SUMMER 2008 - ASSIGNMENT #11

Taylor's theorem

(1) Compute the linear approximation of $\sqrt[3]{10}$. Estimate the error using Lagrange's formula for the remainder.

(2) In this exercise we compute Taylor's polynomial at 0 of degree n for $f(x) = \frac{1}{1-x}$ and the remainder formula.
   (a) Find $f'(x), f''(x)$ and $f^{(3)}(x)$
   (b) Prove by induction that $f^{(n)}(x) = n!(1-x)^{-(n+1)}$
   (c) Find $a_k = \frac{f^{(k)}(0)}{k!}$ for $0 \leq k \leq n$. These are the coefficients in the Taylor polynomial.
   (d) Find $T_n(x)$.
   (e) Find $R_n(x)$ using the formula $1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$

(3) In this exercise we compute Taylor's polynomial at 0 of degree n for $f(x) = \ln(1+x)$ and the remainder formula.
   (a) Find $f'(x), f''(x)$ and $f^{(3)}(x)$ for $x > -1$ (i.e. those x where $\ln(x+1)$ is defined).
   (b) Prove by induction that $f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{(1+x)^n}$
   (c) Find $f^{(k)}(0)$ for $0 \leq k \leq n$, and the coefficients of the Taylor polynomial $a_k = \frac{f^{(k)}(0)}{k!}$
   (d) Find $T_n(x)$.
   (e) Find $R_n(x)$ using Lagrange's formula for the remainder. Show that if $|R_n(x)| < \frac{|x|^{n+1}}{n+1}$
   (f) Let $x = \frac{1}{2}$. Estimate $R_n(\frac{1}{2})$. Show that $\lim_{n\to\infty} R_n(\frac{1}{2}) = 0$. For which xs could you use the same argument?

Remark: Part 3f shows that if we consider the series S that we get by taking the limit: $\lim_{n\to\infty} T_n(\frac{1}{2})$ i.e.:

$$S = f(0) + f'(0)\frac{1}{2} + \frac{f''(0)}{2!}\frac{1}{2^2} + \cdots + \frac{f^{(n)}(0)}{n!}\frac{1}{2^n} + \ldots$$

Then $S = f(\frac{1}{2}) = \ln(\frac{3}{2})$.
This is false for points $x_0 > 1$ not only is it not true that $\lim_{n\to\infty} T_n(x_0) \neq \ln(1+x_0)$ but the limit doesn't even exist!

(4) Using your work in problem 3 find $\ln(1.1)$ within an error of 0.001 (first find the degree of the Taylor polynomial such that the remainder is smaller than 0.001).