MATH 3210 - PARTIAL SOLUTION TO ASSIGNMENT #1

3. a) Show that for any $r > 0$ there is a natural number n such that $r < \sqrt{2}n$. (hint: Can set $n = \lceil r \rceil$ i.e. r rounded up, but why?)

   Proof: The archimedean property applied to r gives us a natural n such that $r < n$. $1 < \sqrt{2}$ thus $r = 1 \cdot r < \sqrt{2} \cdot r < \sqrt{2}n$.

   b) Prove that for any $0 < s$ there is a natural number n such that $\frac{1}{\sqrt{2}n} < s$

   Proof: Apply the 3(a) to $r = \frac{1}{s}$. We get an integer n such that $\frac{1}{s} < \sqrt{2}n$ thus $s > \frac{1}{\sqrt{2}n}$.

   c) Prove that for any two numbers $x < y$ there are natural numbers n, m such that $x < \frac{m}{\sqrt{2}n} < y$

   Proof: Here there's a small typo. Either assume that $0 < x < y$ or that n, m can be chosen to be integers (not naturals). We will do the latter i.e. we prove: For all $x < y$ there are integers n, m such that $x < \frac{m}{\sqrt{2}n} < y$.

   Apply the 3(b) to $s = \frac{y-x}{2}$ (since $y > x$ this is a positive number) to get a natural n such that $s > \frac{1}{\sqrt{2}n}$. Thus $\frac{y-x}{2} > \frac{1}{\sqrt{2}n}$. Therefore $\sqrt{2}ny - \sqrt{2}nx > 2$. Since the gap between $\sqrt{2}ny$ and $\sqrt{2}nx$ is larger than 2 there is an integer m strictly between them (here is were we cannot always choose a natural m since y, x might be negative). Thus $\sqrt{2}nx < m < \sqrt{2}ny$. Dividing by $\sqrt{2}n$ we get $x < \frac{m}{\sqrt{2}n} < y$.

   Note: if you had chosen $s = y - x$ you get a gap greater than 1 between $\sqrt{2}ny$ and $\sqrt{2}nx$ but this is not enough to conclude that there is an integer strictly between them because a priori one of them could be the integer itself. And then one of the strict inequalities in $\sqrt{2}nx < m < \sqrt{2}ny$ would be wrong. If you follow this route you must explain why $\sqrt{2}nx, \sqrt{2}ny$ are nessecerily not integers.

   d) Is a number of the form $\frac{m}{\sqrt{2}n}$ (where n, m are natural) rational or irrational - prove your claim.

$\frac{m}{\sqrt{2}n}$ is irrational. Assume it were rational. Then there are integers a, b such that $\frac{m}{\sqrt{2}n} = \frac{a}{b}$ but then $\sqrt{2} = \frac{mb}{na}$ would be rational. Contradicting the theorem we proved in class that $\sqrt{2}$ is irrational.