MATH 1090 SECTION 2 - SUMMER 2007 - PRACTICE FINAL

You have two hours to complete this test. Show all your work. Use your calculator only for computations The use of cell phones and laptops is not allowed.

question	**grade**	**out of**
1		10
2		15
3		10
4		15
5		10
6		15
7		10
8		10
9		15
total		110

Student Number: ____________________

(1) Solve the following equations:

(a) $2x^2 - x - 5 = 0$

solution:

$$x_{1,2} = \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot 2 \cdot (-5)}}{2 \cdot 2} = \frac{1 \pm \sqrt{41}}{4}$$

(b) $\frac{1}{x} + 2 = \frac{x+1}{3}$

solution: We multiply the equation by the product of the denominators: $3x$

$$3 + 6x = x(x+1)$$
$$3 + 6x = x^2 + x$$
$$x^2 - 5x - 3 = 0$$
$$x_{1,2} = \frac{5 \pm \sqrt{(-5)^2 - 4 \cdot 1 \cdot (-3)}}{2} = \frac{5 \pm \sqrt{37}}{2}$$

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(2) Find the domain of definition of the following functions:

(a) $f(x) = \frac{2}{x-2}$

solution: Since the denominator may not be zero: $x \neq 2$, so the domain is $\{x | x \neq 2\}$

(b) $g(x) = \frac{1}{x} + \frac{2}{x-3}$

solution: For $\frac{1}{x}$ to be defined $x \neq 0$.

For $\frac{2}{x-3}$ to be defined $x \neq 3$.

So the domain is $\{x | x \neq 0, 3\}$

(c) $h(x) = \sqrt{x^2 - 8x + 12}$

solution: Let's denote $j(x) = x^2 - 8x + 12$ so $h(x) = \sqrt{j(x)}$. Now square root is not defined whenever the number inside is negative. Hence $h(x)$ is only defined when $j(x)$ is positive or zero. Since $j(x)$ opens upwards this is when $x \leq x_1$ or $x \geq x_2$ where x_1 and x_2 are the x-intercepts of j. See diagram below:

$$x_{1,2} = \frac{8 \pm \sqrt{64 - 4 \cdot 1 \cdot 12}}{2 \cdot 1} = \frac{8 \pm \sqrt{16}}{2} = \frac{8 \pm 4}{2} = 2, 6$$

So the domain is $\{x | x \leq 2 \text{ or } x \geq 6\} = (-\infty, 2] \cup [6, \infty)$.

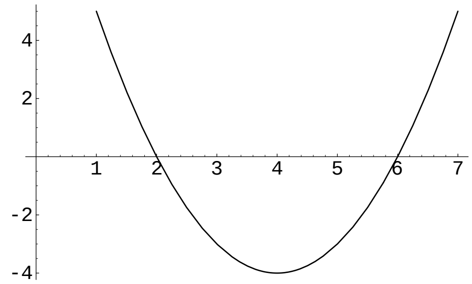

FIGURE 1. The xs that give positive ys are the ones outside the interval between x_1 and x_2

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(3) A businessman must decide whether to lease or buy a car. The cost per month for leasing the car is \$950 which also includes insurance fees. If he rents the car, the cost per mile (for gas, oil and such) is \$0.05. He can purchase a less expensive car for \$11,000 and then the cost per mile becomes \$0.07 per mile. What is the range of miles to drive for which leasing the car is the better option?

solution:

Let x be the number of miles he drives in a year. Cost of leasing $= 950 \cdot 12 + 0.05x$ for the first year

Cost of purchasing $= 11,000 + 0.07x$

If leasing is better then

$$
\begin{aligned}
&950 \cdot 12 + 0.05x < 11,000 + 0.07x \\
&11,400 + 0.05x < 11,000 + 0.07x \\
&400 < 0.02x \\
&20,000 < x
\end{aligned}
$$

He will have to drive more than 20,000 miles a year for leasing to be less expensive than buying.

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(4) (a) If the demand for a product is 300 units then the price per unit is \$50, and if the demand is 500 units then the price is \$40. Find the price as a function of the demand assuming it is linear.

solution: We want to find the price as a function of the demand, so the price is y and the demand is x. We are looking for the line that goes through:

$$(300, 50) \quad (500, 40)$$

The slope of this line will be $a = \frac{40-50}{500-300} = \frac{-10}{200} = -\frac{5}{100} = -0.05$. To find the y-intercept we set $x = 300$, $y = 50$ and $a = -0.05$ in $y = ax + b$ to get:

$$50 = -0.05 \cdot 300 + b$$
$$50 = -15 + b$$
$$65 = b$$

Therefore, the equation of the price as a function of the demand is $y = -0.05x + 65$

(b) Find the revenue as a function of the demand.

solution: Revenue = price · quantity

$R = y \cdot x = (-0.05x + 65)x = -0.05x^2 + 65x$

$R = -0.05x^2 + 65x$

(c) What is the demand that will maximize the revenue?

solution: R is a parabola that opens down. So the maximal value of R will be achieved at the vertex.

$$x_{\text{ver}} = -\frac{b}{2a} = -\frac{65}{2 \cdot (-0.05)} = \frac{65}{0.1} = 650$$

The maximal revenue is achieved when 650 units are sold.

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(5) You are given the choice of investing in option A at a nominal interest rate of 19% compounded quarterly, or in option B yielding a nominal interest rate of 20% compounded semi-annually. Which is the better option?

solution:

We compare the corresponding effective rates.

For option A: 19% compounded quarterly gives a periodic rate of $r = \frac{0.19}{4} = 0.0475$ and $m = 4$ so:

$$r_A^{eff} = (1+r)^m - 1 = (1+0.0475)^4 - 1 = 0.203$$

For option B: 20% compounded semi-annually gives $r = \frac{0.2}{2} = 0.1$ and $m = 2$ so:

$$r_B^{eff} = (1+r)^m - 1 = (1+0.1)^2 - 1 = 0.21$$

Since the effective rate for B is higher, B is the better choice.

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(6) If you put away \$500 in the end of each month in an annuity with APR = 8% compounded quarterly, how long will it take you to save at least \$350,000?

solution:

This is an ordinary annuity since the payments are made at the end of each month.

$$
\begin{aligned}
&R = 500 \\
&r = \tfrac{0.08}{4} = 0.02 \\
&S = 350,000 \\
&S_{\overline{n}|r} = \tfrac{((1+r)^n - 1}{r} = \tfrac{((1+0.02)^n - 1}{0.02} = 50 \cdot (1.02^n - 1) \\
&S = R \cdot S_{\overline{n}|r} \\
&350,000 = 500[50 \cdot (1.02^n - 1)] = 25,000(1.02^n - 1) \\
&14 = 1.02^n - 1 \\
&15 = 1.02^n
\end{aligned}
$$

Now we apply log to both sides and recall the rule that allows us to pull n out of the log to get:

$$
\begin{aligned}
&\log(15) = \log(1.02^n) = n \cdot \log(1.02) \\
&\tfrac{\log 15}{\log 1.02} = n \\
&n = 136.7
\end{aligned}
$$

Since the number of periods of time is an integer number it takes 137 time periods for the money to amount to more than 350,000. Converting to years, it takes $t = \frac{137}{4} = 34.25$ for the annuity to amount to more than 350,000.

(7) Eddy borrowed \$40, 000 from the bank and will pay it off by three payments: \$20, 000 due two months from now, \$10, 000 due three months from now, and a final payment due 6 months from now. How much will the final payment be (at the time of the payment) if the nominal interest rate is 12% compounded monthly?

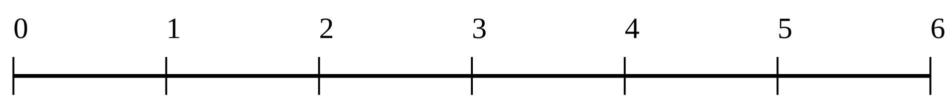
solution:

Call the first payment A, the second payment B, the third payment C, and their sum D. See diagram below.

$D = A + B + C$

$A(2) = 20,000$, $B(3) = 10,000$ and we need to find $C(6)$.

$D(0) = 40,000$.



A	19,605.9		20,000				
B	9705.9			10,000			
C	10,688.2						11,345.7
D	40,000						

FIGURE 2. *The time diagram for problem 7*

We'll use the compounded interest formula to find the present value of A and B and subtract from D:

$r = \frac{0.12}{12} = 0.01$

$A(2) = 20,000$

$A(0) \cdot (1 + 0.01)^2 = 20,000$

$A(0) \cdot 1.0201 = 20,000$

$A(0) = 19,605.9$

$B(3) = 10,000$

$B(0) \cdot (1 + 0.01)^3 = 10,000$

$B(0) \cdot 1.030301 = 10,000$

$B(0) = 9705.9$

$C(0) = D(0) - [A(0) + B(0)] = 40000 - (19605.9 + 9705.9) = 10688.2$

$C(6) = C(0) \cdot (1 + r)^n = 10688.2 \cdot (1 + 0.01)^6 = 11345.7$

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(8) A machine is purchased for \$3000 down and payments of \$250 at the end of every six months for six years. If the interest is at 8% compounded semiannually, find the price of the machine, after 6 years.

solution:

We need to find the future value of the machine in 6 years. We'll use the formula for the amount of an ordinary annuity:

$$S = R \cdot S_{n\neg r}$$

$$R = 250$$

$$r = \frac{0.08}{2} = 0.04$$

$$n = 6 \cdot 2 = 12$$

$$S_{12\neg 0.04} = \frac{(1+0.04)^1 2-1}{0.04} = 15.026$$

$$S = 250 \cdot 15.026 = 3756.45$$

But here's the tricky bit: we must not forget that we paid \$3000 down. To take that into account we must add the future value of the \$3000 in six years. Which comes to:

$$P = 3000$$
$$r = 0.04$$
$$n = 6 \cdot 2 = 12$$
$$A = 3000(1 + 0.04)^{1}2 = 4803.10$$

So the total future dollar value of the machine is:

$$V = 4803.10 + 3756.45 = 8559.55$$

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(9) Solve the following systems of equations. If there's no solution, write: no solution. If there are infinitely many solutions, find the parametric solution.

(a)

$$\begin{cases} x - y - 2z &= -8 \\ -x + 2y + 6z &= 11 \\ 2x + 5z &= -7 \end{cases}$$

solution:

The corresponding matrix to this system is

$$\left(\begin{array}{ccc|c} 1 & -1 & -2 & -8 \\ -1 & 2 & 6 & 11 \\ 2 & 0 & 5 & -7 \end{array}\right) \xrightarrow{\overline{R_2 + R_1 \to R_2}} \left(\begin{array}{ccc|c} 1 & -1 & -2 & -8 \\ 0 & 1 & 4 & 3 \\ 2 & 0 & 5 & -7 \end{array}\right) \xrightarrow{\overline{R_3 - 2R_1 \to R_3}}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -2 & -8 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 9 & 9 \end{array}\right) \xrightarrow{\overline{R_1 + R_2 \to R_1}} \left(\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 9 & 9 \end{array}\right) \xrightarrow{\overline{R_3 - 2R_2 \to R_3}}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & -8 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow{\overline{R_1 - 2R_3 \to R_1}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -11 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow{\overline{R_2 - 4R_3 \to R_2}}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

This system has a unique solution: $x = -11$, $y = -9$ and $z = 3$.

(b)

$$\begin{cases} x - 3y & = & 3 \\ 2x - 6y + 2z & = & 14 \end{cases}$$

solution:

$$\left(\begin{array}{ccc|c} 1 & -3 & 0 & 3 \\ 2 & -6 & 2 & 14 \end{array}\right) \xrightarrow[R_2 - 2R_1 \to R_2]{} \left(\begin{array}{ccc|c} 1 & -3 & 0 & 3 \\ 0 & 0 & 2 & 8 \end{array}\right) \xrightarrow[\frac{1}{2}R_2 \to R_2]{}$$

$$\left(\begin{array}{ccc|c} 1 & -3 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right)$$

The system corresponding to this matrix is:

$$\begin{cases} x - 3y & = & 3 \\ z & = & 4 \end{cases}$$

The parametric solution is:

$$\begin{cases} x & = & 3 + 3r \\ y & = & r \\ z & = & 4 \end{cases}$$