MATH 1090 SECTION 2 - SUMMER 2007 - PRACTICE MIDTERM

You have two hours to complete this test. Show all your work. Calculators are NOT allowed.
Each of the following 8 questions is worth 15 points. Together, they are worth a total of 120 points. The maximum grade is 100 points. You may do part of a problem to get partial credit.

Student Number: ______________________________

(1) Solve: $\dfrac{4}{x-3}+\dfrac{2}{3}=\dfrac{6}{5}-\dfrac{12}{15-5x}$

Solution:

$$\frac{4}{x-3}+\frac{2}{3}=\frac{6}{5}-\frac{12}{15-5x}$$

$$\frac{4}{x-3}+\frac{12}{15-5x}=\frac{6}{5}-\frac{2}{3} \tag{1}$$

Lets simplify the left hand side:

$$\frac{4}{x-3}+\frac{12}{15-5x}=\frac{4}{x-3}+\frac{12}{5(3-x)}$$

Now $(3-x)=-(x-3)$ so:

$$\frac{4}{x-3}-\frac{12}{5(x-3)}=\frac{20}{5(x-3)}-\frac{12}{5(x-3)}=\frac{12}{5(x-3)}=\frac{8}{5(x-3)}$$

Now we'll work on the right hand side of equation 1:

$$\frac{6}{5}-\frac{2}{3}=\frac{18}{15}-\frac{10}{15}=\frac{8}{15}$$

So equation 1 becomes:

$$\frac{8}{5(x-3)}=\frac{8}{15}$$
$$\frac{1}{5(x-3)}=\frac{1}{15}$$
$$\frac{1}{x-3}=\frac{1}{3}$$
$$3=x-3$$
$$x=6$$

You can easily check this solution by plugging in to the original equation.

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(2) Solve: $\frac{1}{2}x^2 - \frac{2}{3}x + \frac{2}{5} = 0$

Solution:

$$x_{1,2} = \frac{-\left(\frac{2}{3}\right) \pm \sqrt{\left(-\frac{2}{3}\right)^2 - 4\left(\frac{1}{2}\right)\left(\frac{2}{5}\right)}}{2\frac{1}{2}} = \frac{-\left(\frac{2}{3}\right) \pm \sqrt{\frac{4}{9} - \frac{8}{10}}}{1}$$

But since $\frac{4}{9} < \frac{8}{10}$ the expression inside the square root is negative, so there's no solution to this equation.

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(3) (a) Find the slope-intercept equation of the line which passes through $(2,9)$ and is parallel to $y = 2x + 1$.

Solution: The slope intercept form is: $y = ax + b$. Since this line is parallel to $y = 2x + 1$, then $a = 2$. It passes through $(2,9)$ so $9 = 2 \cdot 2 + b$. Thus $b = 5$ and the equation is $y = 2x + 5$.

(b) Find the line perpendicular to the line in (3a) and which passes through $(5,2)$.

Solution: This new line has the form $y = cx + d$ and is perpendicular to $y = 2x + 5$. Hence $c = -\frac{1}{2}$. Since it passes through $(5,2)$ we get $2 = -\frac{1}{2}5 + d$ hence $d = 4\frac{1}{2}$, and the equation is: $y = -\frac{1}{2}x + 4\frac{1}{2}$.

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(4) The cost per unit depends on the number of units which are manufactured. Suppose it costs \$150 to manufacture 300 units and \$160 to manufacture 400 units.

(a) Find the cost per unit in each case.

Solution:

\$150 to manufacture 300 units $\Rightarrow$ the cost per unit is $\frac{150}{300} = 0.5$ dollars per unit.

\$160 to manufacture 400 units $\Rightarrow$ the cost per unit is $\frac{160}{400} = \frac{4}{10} = 0.4$ dollars per unit.

(b) Suppose the cost per unit is a linear function of the number of units manufactured. Find the slope - intercept form of this function.

Solution:

If x=300 units then y=0.5 is the cost per unit.

If x=400 units then y=0.4 is the cost per unit.

So the graph of this linear function is a line which passes through $(300, 0.5)$ and $(400, 0.4)$. The equation corresponding to this line is:

$$a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0.4 - 0.5}{400 - 300} = \frac{-0.1}{100} = -0.001$$

We get b by solving:

$$0.5 = -0.001 \cdot 300 + b$$
$$0.5 = -0.3 + b$$
$$b = 0.8$$

So the equation is: $y = -0.001 \cdot x + 0.8$ and this is also the cost per unit as a function of the number of units.

(c) Find the cost per unit when 350 units are manufactured.

Solution: Set $x = 350$ then $y = -0.001 \cdot 350 + 0.8 = -0.35 + 0.8 = 0.45$ dollars per unit.

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(5) A gas station uses the following demand function: $p = 8 - 0.2x$ Where x is the number of (thousands of) gallons sold per day and p is the price per gallon (in dollars). Find the number of gallons to sell to get a maximum revenue. Find the optimal price per gallon and the maximum revenue.

Solution:

Let R denote the revenue. Then

$$R = \text{price} \cdot \text{units} = (8 - 0.2 \cdot x)x = -0.2x^2 + 8x$$

This parabola opens downward, so the vertex is a maximum for the revenue function. The coordinates of the vertex:

$$x_v = \frac{-b}{2a} = \frac{-8}{2 \cdot (-0.2)} = \frac{-8}{-0.4} = 20$$

The maximum revenue will be attained when 20 thousands of gallons are sold. The maximum revenue will be:

$$R_v = R(x_v) = (8 - 0.2 \cdot 20) \cdot 20 = (8 - 4) \cdot 20 = 4 \cdot 20 = 80$$

That is 80 thousands of dollars or $\$80,000$. The price per gallon will be

$$P = 8 - 0.2 \cdot x_v = 8 - 0.2 \cdot 20 = 8 - 4 = 4$$

dollars per gallon.

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(6) A company's margin of profit is: $\frac{\text{net income}}{\text{net sales}}$.

(a) A home business for quilts sold 400 quilts last year at a price of \$13 a unit. If z denotes last year's net income, express last year's margin of profit in terms of z.

Solution: Net sales amounts to price times units sold, so:

$$M_l = \frac{z}{400 \cdot 13}$$

(b) This year, the price increased by \$2 and still 400 quilts were sold. The net income grew by \$100. Express this year's margin of profit in terms of z. (Hint: First write this year's income in terms of z).

Solution: This year's net income is $z + 100$

This year's price per unit is $13 + 2 = 15$

The number of units sold is 400. Therefore:

$$M_t = \frac{z + 100}{400 \cdot 15}$$

(c) Suppose this year's margin of profit was 4% higher than last year's. Use (6a) and (6b) to expresses this.

Solution:

$$M_t = M_l + 0.04 \cdot M_l = 1.04 \cdot M_l$$
$$\frac{z + 100}{400 \cdot 15} = 1.04 \cdot \frac{z}{400 \cdot 13}$$

(d) What was last year's net profit? What was this year's net profit?

Solution: Notice that $1.04 = \frac{104}{100} = \frac{26}{25}$. Now we solve:

$$\frac{z+100}{400 \cdot 15} = 1.04 \cdot \frac{z}{400 \cdot 13}$$

$$\frac{z+100}{400 \cdot 15} = \frac{26}{25} \cdot \frac{z}{400 \cdot 13}$$

$$\frac{z+100}{400 \cdot 15} = \frac{2}{25} \cdot \frac{z}{400}$$

$$\frac{z+100}{15} = \frac{2}{25} \cdot z$$

$$\frac{z+100}{3} = \frac{2z}{5}$$

$$5(z+100) = 6z$$

$$5z + 500 = 6z$$

$$500 = z$$

Las year's net profit was 500 dollars, and this year's was 600 dollars.

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(7) Let $f(x) = 2x^2 + 2x - 4$.

(a) Does it open up or down?

Solution: $a > 0$ so it opens up.

(b) Find the vertex and write the equation of the axis of symmetry.

Solution: The vertex: $x_v = \frac{-b}{2a} = \frac{-2}{4} = -\frac{1}{2}$

$y_v = 2\left(-\frac{1}{2}\right)^2 + 2\left(-\frac{1}{2}\right) - 4 = \frac{2}{4} - 1 - 4 = -4\frac{1}{2} = \frac{-9}{2}$

The equation of the axis of symmetry is: $x = x_v = -\frac{1}{2}$

(c) Find the y-intercept. Find the x-intercepts if they exist.

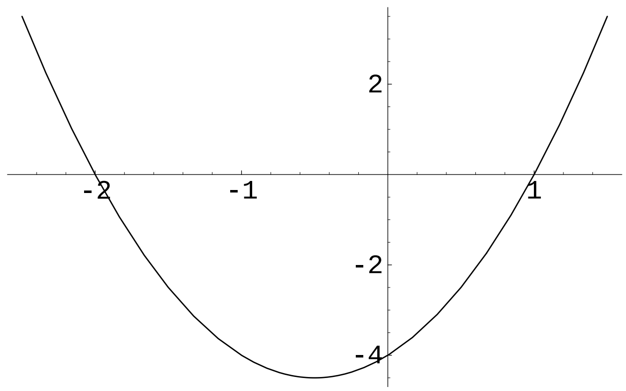
Solution: <u>y-intercept</u>: Set $x = 0$ in the equation and get: $y = 2 \cdot 0^2 + 2 \cdot 0 - 4 = -4$

x-intercept: Set $y = 0$ and solve:

$$x_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4 \cdot (2) \cdot (-4)}}{2 \cdot 2} = \frac{-2 \pm \sqrt{4+32}}{4} = \frac{-2 \pm \sqrt{36}}{4} = \frac{-2 \pm 6}{4} = 1 \text{ or } -2$$

So the intersection points of the graph with the x axis are: $(-2, 0)$ and $(1, 0)$.

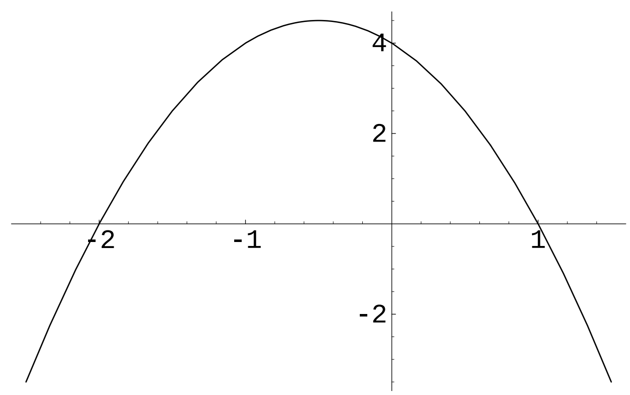
(d) Graph $f(x)$



Solution:

(e) What's the domain of definition of the function $h(x) = \sqrt{-f(x)} = \sqrt{-2x^2 - 2x + 4}$. Sketch your answer on the real line.

Solution: The graph of $-f(x)$ looks like:



because it is the reflection through the x-axis of the graph of f. y is positive only when x is between the two intersection points of the graph with the x axis. So $-f(x)$ is positive only between -2 and 1. So the domain of definition is $[-2, 1]$

(f) Find the vertex of $g(x) = x^2 - 9x + 20$.

Solution: The vertex of g is: $x_v = \frac{-(-9)}{2 \cdot 1} = \frac{9}{2}$

$y_v = (\frac{9}{2})^2 - 9 \cdot (\frac{9}{2}) + 20 = \frac{81}{4} - \frac{81}{2} + \frac{80}{4} = \frac{81-162+80}{4} = -\frac{1}{4}$

(g) How would you need to translate (shift) f's graph so that it's vertex would overlap g's vertex. What is the function of this new graph?

Solution: f's vertex is at $(-\frac{1}{2}, -\frac{9}{2})$ and g's vertex is at $(\frac{9}{2}, -\frac{1}{4})$. So we need to shift f by 5 to the right and $-\frac{1}{4} - (-\frac{9}{2}) = \frac{17}{4}$ up to make them coincide. To shift f by 5 to the right get:

$$f_1(x) = 2(x-5)^2 + 2(x-5) - 4$$

To shift f_1 up by $\frac{17}{4}$ get:

$$f_2(x) = \underbrace{2(x-5)^2 + 2(x-5) - 4}_{f_1(x)} + \frac{17}{4}$$

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(8) Let $f(x) = \frac{1}{x^2}$ and $g(x) = 2x^2 + 3x + 5$.

(a) What are the functions $h(x) = g \circ f(x) = g(f(x))$ and $k(x) = f \circ g(x) = f(g(x))$ do <u>not</u> simplify your answer. Is $h(1) = k(1)$?

Solution:

$$h(x) = g(f(x)) = g\left(\frac{1}{x^2}\right) = 2\left(\frac{1}{x^2}\right)^2 + 3\left(\frac{1}{x^2}\right) + 5$$

and $h(1) = 2\frac{1}{1^2} + 3\frac{1}{1^2} + 5 = 10$

$$k(x) = f(g(x)) = f(2x^2 + 3x + 5) = \frac{1}{(2x^2 + 3x + 5)^2}$$

and $k(1) = \frac{1}{(2 \cdot 1^2 + 3 \cdot 1 + 5)^2} = \frac{1}{10}$ so $h(1) \neq k(1)$

(b) What is $f(x+h)$? What is the difference quotient? $D_1(x,h) = \frac{f(x+h) - f(x)}{h}$? do <u>not</u> simplify your answer.

Solution: $f(x+h) = \frac{1}{(x+h)^2}$ and

$$D_1(x,h) = \frac{f(x+h) - f(x)}{h} = \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

(c) What is $g(x+h)$? What is the difference quotient? $D_2(x,h) = \frac{g(x+h) - g(x)}{h}$? <u>Simplify</u> your answers.

Solution: $g(x+h) = 2(x+h)^2 + 3(x+h) + 5 = 2(x^2 + 2xh + h^2) + 3(x+h) + 5 = 2x^2 + 3x + 5 + 2xh + h^2$ and

$$D_g(x,h) = \frac{g(x+h)-g(x)}{h} = \frac{2x^2+3x+5+2xh+h^2-(2x^2+3x+5)}{h} = \frac{2xh+h^2}{h} = \frac{h(2x+h)}{h} = 2x+h$$