Mapping Class Groups
MSRI, Fall 2007
Day 1, September 6

September 12, 2007

Theme for the course (if there is one):

- Hierarchical structure of mapping class groups

Topics:

- Conjugacy problem for mapping class groups
  - Thurston's trichotomy for mapping classes
  - Conjugacy invariants for reducible mapping classes reveal hierarchical structure.
- Subgroup theorems: Tits alternative, subgroup trichotomy, abelian subgroups.
  - New unifying proofs (joint with M. Handel) reveal hierarchical structure
- Results of Masur and Minsky on curve complexes
  - Hyperbolicity, Hierarchy paths
  - Quasi-distance formula
- ???

Prerequisites (to be reviewed briefly as needed):

- Basics of Teichmüller space
  - Geodesic laminations, measured foliations, compactification of Teichmüller space
  - Thurston's trichotomy for mapping classes.

- Source material:
  - Casson-Bleiler (including the unpublished 2nd volume)
  - Fathi-Laudenbach-Poenaru (translation available from Margalit's web site)

Mapping class group of the torus

- $T = S^1 \times S^1$

- $\pi_1(T, p) \approx H_1(T; \mathbf{Z}) \approx \mathbf{Z}^2$

- $\mathcal{MCG}(T) = \mathrm{Homeo}_+(T) / \mathrm{Homeo}_0(T)$

- The action

$$\mathsf{Homeo}_+(T) \circlearrowleft H_1(T;\mathbf{Z}) \approx \mathbf{Z}^2$$

is trivial on $\mathsf{Homeo}_0(T)$ and so descends to an action

$$\mathcal{MCG}(T) \circlearrowleft H_1(T;\mathbf{Z}) \approx \mathbf{Z}^2$$

which induces a homomorphism

$$\mathcal{MCG}(T) \to \mathsf{Aut}(H_1(T;\mathbf{Z})) \approx \mathsf{SL}(2;\mathbf{Z})$$

- The linear action $\mathsf{SL}(2;\mathbf{Z})$ on $\mathbf{R}^2$ preserves $\mathbf{Z}^2$ and descends to an action on T, inducing a homomorphism

$$\mathsf{SL}(2;\mathbf{Z}) \to \mathsf{Homeo}^+(T) \to \mathcal{MCG}(T)$$

Theorem (Algebraic structure of $\mathcal{MCG}(T)$)**.** *The above two homomorphisms*

$$\mathcal{MCG}(T) \to \mathsf{SL}(2;\mathbf{Z})$$
$$\mathsf{SL}(2;\mathbf{Z}) \to \mathcal{MCG}(T)$$

are inverse isomorphisms of each other.

Action on Teichmüller space. There is an action

$$\text{Homeo}_+(T) \circlearrowleft \{\text{conformal structures on } T\}$$

obtained by pushing a conformal structure forward. This descends to an action

$$\begin{aligned}\mathcal{MCG}(T) = \frac{\text{Homeo}_+(T)}{\text{Homeo}_0(T)} \circlearrowleft \frac{\text{conformal structures}}{\text{Homeo}_0(T)} \\ = \frac{\text{conformal structures}}{\text{isotopy}} \\ = \mathcal{T}(T) \\ = \text{Teichmüller space of T}\end{aligned}$$

By the Uniformization Theorem we have natural isomorphisms

$$\mathcal{T}(T) = \frac{\text{conformal structures}}{\text{isotopy}}$$

$$= \frac{\text{Euclidean structures}}{\text{homothety isotopic to identity}}$$

$$= \frac{\text{Euclidean structures of area 1}}{\text{isometry isotopic to identity}}$$

Picture of Teichmüller space:

- Each Euclidean structure on T is obtained in a unique manner as follows:
  - Start with the standard action of $\mathbf{Z}^2$ on $\mathbf{R}^2$, with fundamental domain $[0,1] \times [0,1]$
  - Conjugate by an orientation preserving linear map $A\colon \mathbf{R}^2 \to \mathbf{R}^2$ which is the identity on the x-axis
  - Result is an isometric action of $\mathbf{Z}^2$ whose fundamental domain $A([0,1] \times [0,1])$ is a parallelogram in the upper half plane with one side being $[0,1] \times 0$.

- The upper right corner $A(1 \times 1)$, regarded as a parameter, gives an isomorphism

$$\mathcal{T}(T) \approx \text{upper half plane}$$

- The isomorphism $\mathcal{MCG}(T) \approx \mathsf{SL}(2,\mathbf{Z})$ gives a commutative diagram of actions

$$\begin{aligned} &\mathcal{MCG}(T) \circlearrowleft \mathcal{T}(T) \\ &\mathsf{SL}(2,\mathbf{Z}) \circlearrowleft \text{upper half plane} \end{aligned}$$

- This action is *not* faithful.

- Kernel is the center $\{\pm \mathrm{Id}\} \approx \mathbf{Z}/2$.

- Induced action of the quotient group

$$\mathsf{PSL}(2;\mathbf{Z}) = \mathsf{SL}(2;\mathbf{Z})/\pm \mathrm{Id} \circlearrowleft \mathcal{T}(T)$$

  is faithful.

- This group of transformations of $\mathcal{T}(T)$ is known as the *modular group* $\mathrm{Mod}(T)$.

Amalgamation structure of $\mathcal{MCG}(T) \approx \mathsf{SL}(2;\mathbf{Z})$

- Fundamental domain for $\mathcal{MCG}(T) \circlearrowleft \mathcal{T}(T)$ ...

- Invariant tree τ for the action ...

- Peripheral lines (horocycles) and nonperipheral lines ...

- Two vertex orbits, Red and Green:
  - Each Red vertex PSL$(2;\mathbf{Z})$ stabilizer $\mathbf{Z}/3$, and SL$(2,\mathbf{Z})$ stabilizer $\mathbf{Z}/6$.
  - Each Green vertex has PSL$(2;\mathbf{Z})$ stabilizer $\mathbf{Z}/2$ and SL$(2,\mathbf{Z})$ stabilizer $\mathbf{Z}/4$.

- One edge orbit:
  - Each edge has $\mathsf{PSL}(2;\mathbf{Z})$ stabilizer 1, and $\mathsf{SL}(2;\mathbf{Z})$ stabilizer the $\mathbf{Z}/2$ central subgroup.

Conclusion by Bass-Serre theory:

$$\mathsf{PSL}(2;\mathbf{Z}) \approx \mathbf{Z}/3 * \mathbf{Z}/2$$
$$\mathsf{SL}(2;\mathbf{Z}) \approx \mathbf{Z}/6 *_{\mathbf{Z}/2} \mathbf{Z}/4$$

Conjugacy classification in $\mathsf{PSL}(2;\mathbf{Z})$ and $\mathsf{SL}(2;\mathbf{Z})$:

- Trace is an conjugacy invariant in $\mathsf{SL}(2;\mathbf{Z})$.

- $|\mathsf{Trace}|$ is a conjugacy invariant in $\mathsf{PSL}(2;\mathbf{Z})$.

- The pre-image of each $\mathsf{PSL}(2;\mathbf{Z})$ conjugacy class is a pair of $\mathsf{SL}(2;\mathbf{Z})$ conjugacy classes, differing by the sign of the trace (except in the case of zero trace).

- Given $\phi \in \mathcal{MCG}(T) \approx \mathsf{SL}(2;\mathbf{Z})$, we consider the following trichotomy:

  - $|\mathrm{Tr}(\phi)| < 2 \iff \phi$ has finite order

  - $|\mathrm{Tr}(\phi)| = 2 \iff \phi$ fixes some simple closed curve

  - $|\mathrm{Tr}(\phi)| > 2 \iff \phi$ is Anosov.

Case 1: Finite order. If $|\text{Tr}(\phi)| < 2$ then ϕ has finite order. There are finitely many such conjugacy classes.

Case 1a: $|\text{Tr}(\phi)| = 0$

- $\iff$ ϕ has order 4, fixing some valence 2 vertex of τ, rotating $\mathcal{T}(T)$ by π around the fixed vertex.

- $\iff$ ϕ leaves invariant some square Euclidean structure on T with rotational holonomy $\pi/4$ or $3\pi/4$. This angle is a *complete* conjugacy invariant.

- Two such conjugacy classes in SL$(2;\mathbf{Z})$, and one in PSL$(2;\mathbf{Z})$.

- Example: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has order 4, fixing the vertex i.

Case 1b: $|\text{Tr}(\phi)| = 1$

- $\iff$ ϕ has order 3 or 6, fixing some valence 3 vertex of τ, rotating $\mathcal{T}(T)$ by $2\pi/3$ or $4\pi/3$ around the fixed vertex.

- $\iff$ ϕ leaves invariant some hexagonal Euclidean structure on T with rotational holonomy $\pi/3$, $2\pi/3$, $4\pi/3$, $5\pi/3$. This angle is a *complete* conjugacy invariant. (The $\mathcal{T}(T)$ rotation angle equals 2 times the Euclidean rotational holonomy).

- Four such conjugacy classes in $\text{SL}(2;\mathbf{Z})$, and two in $\text{PSL}(2;\mathbf{Z})$.

- Example: $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ has order 6, fixing the vertex $\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

Case 2: $|\mathrm{Tr}(\phi)| = 2$

- $\iff$ ϕ fixes some rational number on $\mathbf{R} = \partial\mathcal{T}(T)$, which is the slope of the unique eigenvector of ϕ.

- $\iff$ ϕ preserves some "horocycle" line in the tree $\mathcal{T}$.

- $\iff$ ϕ is a power of a Dehn twist, possibly multiplied by $-\mathrm{Id}$ (ϕ could be $\pm\mathrm{Id}$).

- $\iff$ ϕ is conjugate to a matrix of the form $\pm\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$.

- Moreover, ϕ is conjugate to a *unique* matrix of the form $\pm\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$, and the integer n together with the sign of the trace are *complete* conjugacy invariants.

Case 3: Anosov. If $|\mathrm{Tr}(\phi)| > 2$

- $\iff$ ϕ fixes a pair of irrational numbers on $\mathbf{R} \cup \{\infty\} = \partial\mathcal{T}(T)$, one the slope of an expanding eigenvector, one the slope of a contracting eigenvector, with respective eigenvectors $\lambda > 1$, $\lambda^{-1} < 1$.

- $\iff$ ϕ preserves a non-horocyclic line ℓ in the tree τ, and a fellow travelling geodesic γ in $\mathcal{T}(T)$, translating along γ a distance $\log(\lambda)$. (ideal endpoings of ℓ or of γ are the fixed points in $\partial\mathcal{T}(T)$

- $\iff$ ϕ is represented by an Anosov homeomorphism of the torus: there exists
  - Euclidean structure μ on T
  - Pair of μ-orthogonal foliations, $\mathcal{F}^u$ ("unstable" or "horizontal" foliation), and $\mathcal{F}^s$ ("stable" or "vertical" foliation)
  - $\lambda > 1$

  such that

  - ϕ preserves $\mathcal{F}^u$, stretching leaves by factor λ
  - ϕ compresses $\mathcal{F}^s$, compressing leaves by factor λ

Anosov conjugacy classification, method 1.

- In τ, consider the invariant line ℓ, oriented in the direction of translation . . .

- Each time ℓ passes a valence 3 vertex, it turns L or R . . .

- Get a bi-infinite sequence of L's and R's, on which ϕ acts.

- Quotient of this sequence under ϕ action is an oriented loop of L's and R's of even length

$$\left(p_i \,\middle|\, i \in \mathbf{Z}/2k\right), \quad p_i \in \{L, R\}$$

- This loop (up to cyclic permutation), and the trace, is a complete conjugacy invariant of ϕ.

- Set $M_L = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $M_R = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ (or the other way around. . . I'm not sure. . .).

- If $\mathrm{Tr}(\phi) > 2$, the conjugacy class of ϕ is represented by the following positive matrices *and no other positive matrices*:

  $$M_{p_1} \cdot M_{p_2} \cdot \ldots \cdot M_{p_{2k-1}} \cdot M_{p_{2k}}$$

  $$M_{p_2} \cdot M_{p_3} \cdot \ldots \cdot M_{p_{2k}} \cdot M_{p_1}$$

  and other cyclic conjugates

Next time: We use dynamical systems — the stable and unstable foliations — to give another description of the Anosov conjugacy classification, one which will generalize to all finite type surfaces.