# Mapping Class Groups
# MSRI, Fall 2007
# Day 10, November 29

December 6, 2007

**Recap: three theorems about subgroups - Ivanov; McCarthy + Birman/McCarthy/Lubotzky**

$\forall$ finite type surface $S$ and subgroup $G < \mathcal{MCG}(S)$,

**Tits Alternative** Either $G$ contains $F_n$ with $n \geq 2$, or $G$ contains a finite index abelian subgroup.

**Subgroup trichotomy** Either $G$ is finite, or $G$ has a reducing system, or $G \ni$ a pseudo-Anosov element.

**Classification of abelian subgroups** $G$ abelian $\implies$
$\exists$ essential subsurface $F = F_1 \cup \cdots \cup F_K \subset S$ and $\Phi_1, \ldots, \Phi_K \in \text{Homeo}_+(S)$ s.t. $\Phi_k \mid S - F_k = \text{Id}$, and:

- $F_k = \text{annulus} \implies \Phi_k \mid F_k$ is a Dehn twist power
- $F_k \neq \text{annulus} \implies \Phi_k \mid F_k$ is pseudo-Anosov.
- $G$ has a finite index subgroup in $\langle \Phi_1 \rangle \oplus \ldots \oplus \langle \Phi_K \rangle$ (See figure for example of where we need to pass to a finite index subgroup).

We are proving all three of these theorems as applications of a single Omnibus Subgroup Theorem (statement and application a little later; proof next time).

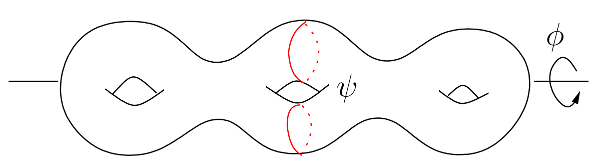


Figure 1: Suppose $G =< \phi, \psi >$. $\phi$ has order 2 and $\psi$ is a pseudo-Anosov on the twice punctured torus. $< \phi >$ is an index 2 subgoup of $G$.

**Last time:**

- Started proof of Tits alternative when $G$ contains a pseudo-Anosov element.
- Source–Sink dynamics: Action of a pseudo-Anosov $\phi \in \mathcal{MCG}(S)$ on $\mathbf{P}\mathcal{M}F$ has "source–sink" or "north–south" dynamics:
  $\exists \xi_\phi^\pm \in \mathbf{P}\mathcal{M}F$, such that $\xi_\phi^+ \neq \xi_\phi^-$ and such that $\forall \eta \in \mathbf{P}\mathcal{M}F$,
  - If $\eta \neq \xi_\phi^+$ then $\lim_{n\to+\infty} \phi^n(\eta) = \xi_\phi^-$
  - If $\eta \neq \xi_\phi^-$ then $\lim_{n\to+\infty} \phi^{-n}(\eta) = \xi_\phi^+$

**Next: Stabilizers of arational measured foliations**

A rational foliation is one that contains closed leaves or a loop of saddle connections or a path of saddle connections between punctures. A foliation that is not rational is an arational foliation.

$\forall$ arational $\mathcal{F} \in \mathcal{M}F$ with projective class $\xi \in \mathbf{P}\mathcal{M}F$.

$\text{Stab}(\mathcal{F})$ = stabilizer of $\mathcal{F}$ under action of $\mathcal{MCG}(S)$ on $\mathcal{M}F$

$\text{Stab}(\xi)$ = stabilizer of $\xi$ under action of $\mathcal{MCG}(S)$ on $\mathbf{P}\mathcal{M}F$.

Define the "log stretch" homomorphism

$$\text{Stab}(\xi) \xrightarrow{\ell_\xi} \mathbf{R}$$

$$\psi(\mathcal{F}) = \exp(\ell_\xi) \cdot \mathcal{F}$$

Note that

$$\ker(\ell_\xi) = \text{Stab}(\mathcal{F})$$

**Theorem (Stretch Theorem).** $\text{image}(\ell_\xi)$ *is discrete and* $\ker(\text{Stab}(\xi)) = \text{Stab}(\mathcal{F})$ *is finite.*
$\implies$ $\text{Stab}(\xi)$ *is finite or virtually cyclic.*

*Proof that* $\text{image}(\ell_\xi)$ *is discrete:*

- $\ell_\xi(\psi) \neq 0 \iff \psi$ is pseudo-Anosov and $\xi$ =projective class of $\mathcal{F}_\psi^s$ or $\mathcal{F}_\psi^u$ (see foe example FLP)
  $\implies \lambda_\psi = \exp|\ell_\xi(\psi)|$ is the stretch factor.

- $\lambda_\psi$ is the Perron-Frobenius eigenvalue of a non-negative integer matrix of bounded size.
- The set of such numbers is discrete.

$\diamondsuit$

*Proof that* Stab($\mathcal{F}$) *is finite:* (A piece of Thurston's original proof of the trichotomy for elements of $\mathcal{MCG}(S)$. Proof shows that Stab($\mathcal{F}$) is represented by a finite subgroup of $\text{Homeo}_+(S)$.)

- Pick an actual measured foliation $F$ in the class $\mathcal{F}$.
- May assume $F$ has no saddle connections (collapse them if there are any; this uses that $F$ is arational).
- With this assumption, $F$ is unique up to isotopy (not just up to Whitehead equivalence).
- Each $\psi \in \text{Stab}(\mathcal{F})$ is represented by $\Psi \in \text{Homeo}_+(S)$ such that $\Psi(F) = F$ (preserving measure!)
  - because $F$ is unique up to isotopy (Notice that if $F'$ has saddle connections then $\psi(F')$ might not be isotopic to $F'$. See figure for an example).
- The action of $\Psi$ on leaves of $F$ depends only on $\psi$.
- Proof (pictures)
  - Suppose $\Psi, \Psi' \in \text{Homeo}_+(S)$ represent $\psi$, and $\Psi(F) = \Psi'(F) = F$, and $L$ is a leaf.
  - Lift to universal cover $\widetilde{S} = \mathbf{H}^2$ so that $\widetilde{\Psi}, \widetilde{\Psi}'$ have same action on the boundary.
  - $\implies \widetilde{\Psi}(\partial\widetilde{L}) = \widetilde{\Psi}'(\partial\widetilde{L})$

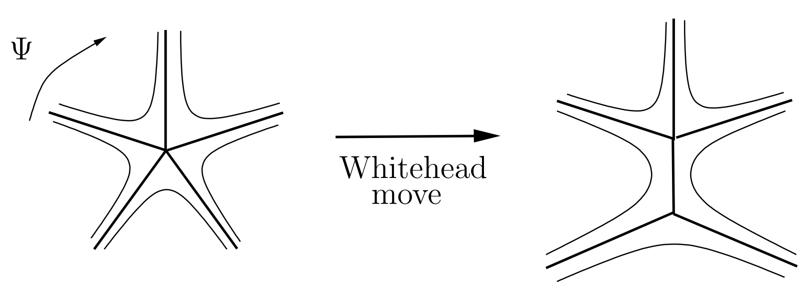


Figure 2: Suppose $\Psi$ acts on $F$ on the left by rotating its separatrices - $F$ is preserved under $\Psi$, but the foliation on the right is not preserved by $\phi$ since it would not preserve the grouping of the branches.

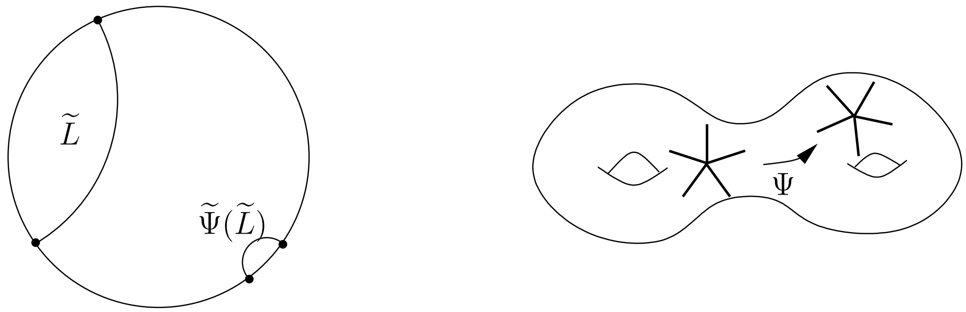


Figure 3: $\widetilde{\Psi}, \widetilde{\Psi}'$ take $\widetilde{L}$ to the same leaf of $\widetilde{F}$ since they act on $\partial\widetilde{S}$ in the same way. They also permute the separatrices and singularities in the same way.

  - $\implies \widetilde{\Psi}(\widetilde{L}) = \widetilde{\Psi}'(\widetilde{L})$
  - $\implies \Psi(L) = \Psi'(L)$

- Follows that $\mathrm{Stab}(\mathcal{F})$ acts on the singularities and the separatrices of $F$.

**Remains to prove:** If $\Psi(F) = F$ and if $\Psi$ preserves the singularities and separatrices of $F$ then $\Psi$ is isotopic to the identity.

- Pick a singularity $s$, a sector at $s$, and a positive length transversal $\alpha$ in that sector. (PICTURE)
- For each $r \in (0, \mathrm{measure}(\alpha)]$ let $\alpha_r$ be the subsegment of transverse measure $r$.
- $\Psi$ preserves sectors, so $\Psi(\alpha)$ is in the same sector (PICTURE).
- Both $\alpha_r$ and $\Psi(\alpha_r)$ have transverse measure $r$, both have endpoint $s$, both are in the same sector at $s$.
- There exists $r$ such that $\alpha_r$ and $\Psi(\alpha_r)$ are isotopic along leaves rel $s$.
- Alter $\Psi$ by this isotopy.
- After this isotopy, $\Psi \bigm| \alpha_r = \mathrm{Id}$.
- Using $\alpha_r$, decompose $S$ into rectangles (This uses the arationality of $F$ - every half leaf is dense).
- $\Psi$ is the identity on each vertical side, preserves each horizontal side, and preserves each rectangle.
- Isotope $\Psi$ to the identity on each horizontal side.
- Isotope $\Psi$ to the identity on each rectangle.
- $\Psi$ is now the identity.

$\diamondsuit$

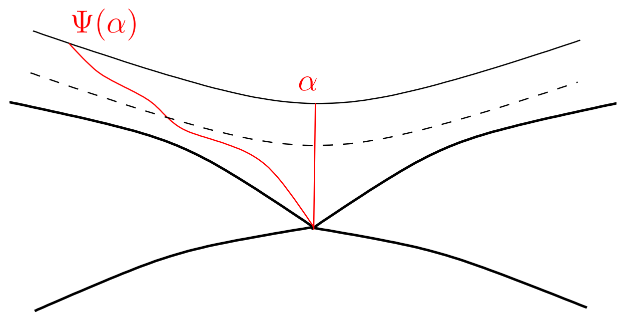


Figure 4: $\Psi(\alpha)$ is in the same sector as $\alpha$ but there might be some topology between $\alpha$ and $\Psi(\alpha)$ so choose a subsegment $\alpha_r$ where $\Psi(\alpha_r)$ is isotopic to $\alpha_r$. Isotope $\Psi$ to preserve $\alpha_r$. Use the first return map of $\alpha_r$ to form a rectangle decomposition to conclude that $\Psi$ is isotopic to the identity.

**Corollaries to Stretch Theorem:**

Given a pseudo-Anosov $\phi \in \mathcal{MCG}(S)$, notation:
$\xi_\phi^- \in \mathbf{P}\mathcal{M}F$ is the source.

$\xi_\phi^+ \in \mathbf{P}\mathcal{M}F$ is the sink.

$\mathcal{F}_\phi^- = \mathcal{F}_\phi^s \in \mathcal{M}F$ is (the class of) the stable measured foliation, whose projective class is $\xi_\phi^-$

$\mathcal{F}_\phi^+ = \mathcal{F}_\phi^u \in \mathcal{M}F$ is (the class of) the unstable measured foliation, whose projective class is $\xi_\phi^+$.

NOTE: the stable and unstable measured foliations are *arational*, so the Stretch Theorem applies.

**Corollary 1.** *For any pseudo-Anosov $\phi \in \mathcal{MCG}(S)$ with source $\xi_\phi^-$ and sink $\xi_\phi^+ \in$ $\mathbf{P}\mathcal{M}F$ we have:*

$$\mathrm{Stab}(\xi_\phi^-) = \mathrm{Stab}(\xi_\phi^+)$$

**Step 1:** $\mathrm{Stab}(\xi_\phi^+)$ is virtually cyclic,
$\implies \langle\phi\rangle < \mathrm{Stab}(\xi_\phi^+)$ has finite index.

**Step 2:** Given $\psi \in \mathrm{Stab}(\xi_\phi^+)$, the mapping class $\psi\phi\psi^{-1}$ is pseudo-Anosov with source $\psi(\xi_\phi^-)$ and sink $\xi_\phi^+$
$\implies \langle\psi\phi\psi^{-1}\rangle < \mathrm{Stab}(\xi_\phi^+)$ has finite index.

**Step 3:** The two mapping classes $\phi$, $\psi^{-1}\phi\psi \in \mathrm{Stab}(\xi_\phi^+)$ have expansion factors $> 1$ so have positive powers which are equal (since $\mathrm{Stab}(\xi_\phi^+)$ is virtually cyclic)

$$(\psi\phi\psi^{-1})^m = \phi^n$$

**Remark.** In particular $\phi^n$ and $(\psi\phi\psi^{-1})^m$ have the same source.

**Step 4:** $\implies$ $\phi$, $\psi^{-1}\phi\psi$ have the same source (because the source and sink of a pseudo-Anosov homeomorphism don't change under positive powers)

$$\psi(\xi_\phi^-) = \xi_\phi^-$$

**Corollary 2.** *For any two pseudo-Anosov mapping classes $\phi_1, \phi_2 \in \mathcal{MCG}(S)$, the pairs $\xi^\pm_{\phi_1}$, $\xi^\pm_{\phi_2}$ are either equal or disjoint.*

*Proof.* Assume they are not disjoint. Replacing $\phi_1$ and/or $\phi_2$ by its inverse, may assume

$$\xi^+_{\phi_1} = \xi^+_{\phi_2}$$

$\implies \phi_2 \in \mathrm{Stab}(\xi^+_{\phi_2}) = \mathrm{Stab}(\xi^+_{\phi_1}) = \mathrm{Stab}(\xi^-_{\phi_1})$
$\implies \phi_2(\xi^-_{\phi_1}) = \xi^-_{\phi_1}$
$\implies \xi^-_{\phi_2} = \xi^-_{\phi_1}$ because of source-sink dynamics.

$\diamondsuit$

*Proof of Tits Alternative with a pseudo-Anosov element.* Suppose the subgroup $G < \mathcal{MCG}(S)$ has a pseudo-Anosov element $\phi$.

**Case 1:** $G$ preserves the subset $\xi^\pm_\phi$ (This is the case where $G$ is virtually cyclic). The stablizer of this subset contains $\mathrm{Stab}(\xi^+_\phi)$ with index at most 2, which contains the infinite cyclic group $\langle\phi\rangle$ with finite index.

**Case 2:** $G$ does not preserve the subset $\xi^\pm_\phi$.
Choose $\psi$ so that $\psi(\xi^\pm_\phi) \neq \xi^\pm_\phi$.
By the corollary, $\psi(\xi^\pm_\phi)$ and $\xi^\pm_\phi$ are disjoint.
Let $\phi' = \psi\phi\psi^{-1}$, so $\xi^\pm_{\phi'} = \psi(\xi^\pm_\phi)$ and $\xi^\pm_\phi$ are disjoint.
Play ping-pong: on a compact space, if two homeomorphisms have source-sink dynamics with disjoint source-sink pairs, then some powers freely generate an $F_2$ subgroup. $\diamondsuit$

**Recap: statement of Omnibus Subgroup Theorem**
Consider $\phi \in G$.
$\mathcal{C}_\phi$ = canonical reducing system
($\neq \emptyset \iff \phi$ is infinite order and reducible).
$N_\phi$ = regular neighborhood of $\mathcal{C}_\phi$.
$\mathcal{A}_\phi$ = *active subsurface* of $\phi$, defined to be the union of:

- Components of $S - N_\phi$ on which the first return mapping class is pseudo-Anosov
- Components $A$ of $N_\phi$ such that the components of $S - N_\phi$ on either side of $A$ have first return mapping class of finite order.

Features of the active subsurface $\mathcal{A}_\phi$:
• $\mathcal{A}_\phi$ is an essential subsurface.
• No annulus component of $\mathcal{A}_\phi$ is isotopic into a distinct component.
• $\mathcal{A}_\phi = \emptyset$ if and only if $\phi$ has finite order
• $\mathcal{A}_\phi = S$ if and only if $\phi$ is pseudo-Anosov.

**Theorem 3** (Omnibus Subgroup Theorem (Handel-M))**.** *Every subgroup contains an element whose active subsurface is maximal.*

*More precisely, for every subgroup $G < \mathcal{MCG}(S)$ there exists $\phi \in G$ such that for every $\psi \in G$, the subsurface $\mathcal{A}_\psi$ is isotopic into the subsurface $\mathcal{A}_\phi$.*

We shall refer to $\phi$ as a *maximally active* element of $G$.

**Last time proved:**

Omnibus Subgroup Theorem $\implies$ Subgroup Trichotomy.
**Important lemma in the proof:** If $\phi \in G$ is maximally active then $\psi(\mathcal{A}_\phi) = \mathcal{A}_\phi$ for all $\psi \in G$.
**Corollary:** If $\phi, \psi \in G$ are both maximally active then $\mathcal{A}_\phi, \mathcal{A}_\psi$ are isotopic.
We may therefore define
$\mathcal{A}_G$ = active subsurface of the subgroup $G = \mathcal{A}_\phi$ for any maximally active $\phi \in G$.
$\mathcal{A}_G$ is well-defined up to isotopy.

**Next:** Reformulate and (very quickly) prove:

Omnibus Subgroup Theorem $\implies$ Tits Alternative and Classification of Abelian Subgroups.

**Definition:** Given an infinite order, irreducible subgroup $G < \mathcal{MCG}(S)$ which has a pseudo-Anosov element, either:

$G$ **is elementary** meaning that $G$ has a virtually cyclic pseudo-Anosov subgroup of fintie index; or

$G$ **is nonelementary** meaning that $G$ has an $F_2$ subgroup.

Given $G < \mathcal{MCG}(S)$ which is infinite order and reducible. $\implies$ $\mathcal{A}_G$ is nonempty and proper.

$G$ acts on the set of components $F$ of $\mathcal{A}_G$ and of $S - \mathcal{A}_G$.

Let $G_0$ = kernel of this action, a finite index subgroup of $G$ that preserves each $F$.

Let $G_0 \to \mathcal{MCG}(F)$ be the restriction homomorphism.
Denote its image by $G_0 \mid F$.

Let $G_1 = \bigcap_F \ker\big(G_0 \to \mathcal{MCG}(F) \to Out(H_1(F; \mathbf{Z}/3))\big)$

$\implies G_1 =$ finite index subgroup of $G$ and
$G_1 \bigm| F$ is torsion free for each $F$

**List of special cases for $G_1 \bigm| F$:**

- $F =$ component of $S - \mathcal{A}_G \implies G_1 \bigm| F$ is trivial (because it is finite and torsion free).
- $F =$ nonannulus component of $\mathcal{A}_G \implies G_1 \bigm| F$ is irreducible.
  - if $G_1 \bigm| F$ is elementary then it is an infinite cyclic pseudo-Anosov subgroup.
  - if $G_1 \bigm| F$ is nonelementary then it contains an $F_2$ (by the special case of the Tits alternative).
- $F =$ annulus component of $\mathcal{A}_G \implies G_1 \bigm| F$ is an infinite cyclic subgroup generated by a Dehn twist power.

**Theorem 4** (Tits Alt. + Class. of Abel Subgps)**.** *Exactly one of the following is true:*

1. *There exists a nonannulus component $F$ of $\mathcal{A}_G$ such that the image of $G_1 \to \mathcal{MCG}(F)$ is nonelementary.*
   *$\implies G_1$ contains an $F_2$ subgroup.*
2. *For each nonannulus component $F$ of $\mathcal{A}_G$ the image of $G_1 \to \mathcal{MCG}(F)$ is elementary.*
   *$\implies G_1$ is free abelian and satisfies the conclusions of the Classification of Free Abelian Subgroups.*

*Proof of (1).* The image of the homomorphism $G_1 \to \mathcal{MCG}(F)$ contains an $F_2$.
There is a homomorphic section from this $F_2$ back to $G_1$. ♢

*Proof of (2).* Follows immediately from the list of special cases for $G_1 \bigm| F$. ♢