Mapping Class Groups
MSRI, Fall 2007
Day 11, December 6

December 12, 2007

Theorem 1 (Omnibus Subgroup Theorem (Handel-M)). *For every subgroup $G < \mathcal{MCG}(S)$ there exists $\phi \in G$ such that for every $\psi \in G$, the subsurface $\mathcal{A}_\psi$ is isotopic into the subsurface $\mathcal{A}_\phi$.*

Remark. The real motivation for this theorem is as a warmup to understanding $\mathrm{Out}F_n$. Bestvina-Feign-Handel proved the Tits alternative theorem for $\mathrm{Out}F_n$. Feign-Handel proved a version of the classification of abelian subgroups for $\mathrm{Out}F_n$. And a version of the Omnibus Subgroup theorem led to a proof of the subgroup trichotomy theorm for $\mathrm{Out}F_n$ by M-Handel.

Proof:

- Essential subsurfaces are partially ordered by inclusion up to isotopy.
- Any strictly increasing chain has bounded length.
- $\implies \exists \phi \in G$ such that $\mathcal{A}_\phi$ is "weakly maximal" meaning:
  - $\forall \psi \in G$, if $\mathcal{A}_\phi$ is isotopic into $\mathcal{A}_\psi$ then $\mathcal{A}_\phi, \mathcal{A}_\psi$ are isotopic to each other.
- Shall prove that a weakly maximal ϕ is maximally active:
  - $\forall \psi \in G$, $\mathcal{A}_\psi$ is isotopic into $\mathcal{A}_\phi$.
- Let $B =$ essential subsurface filled by $\mathcal{A}_\phi, \mathcal{A}_\psi$.
  - Pull $\partial\mathcal{A}_\phi, \partial\mathcal{A}_\psi$ tight with respect to each other.
    - Isotopic components are *equal*
    - No intersection component is an annulus *unless* it is an annulus component of both
    - Otherwise, boundary components intersect efficiently.
  - $B = \mathcal{A}_\phi \cup \mathcal{A}_\psi \cup$ (components of $S - (\mathcal{A}_\phi \cup \mathcal{A}_\psi)$ which are discs with ≤ 1 puncture)

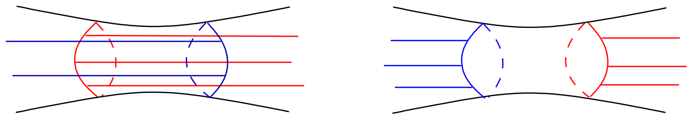

Figure 1: Pulling A_ϕ and A_ψ tight.

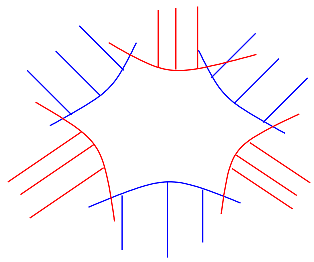

Figure 2: Add discs or punctured discs to $A_\phi \cup A_\psi$

Reduction: Shall produce $\phi^m\psi^n \in G$ whose active subsurface is the surface B filled by $\mathcal{A}_\phi, \mathcal{A}_\psi$.

This finishes the proof, because:

- $\mathcal{A}_\phi \subset B$, so weak maximality $\implies$ $\mathcal{A}_\phi = B$.
- $\mathcal{A}_\psi \subset B$, so $\mathcal{A}_\psi \subset \mathcal{A}_\phi$ as desired.

How to produce the desired $\phi^m\psi^n$:

- May choose Φ, Ψ representing ϕ, ψ to be the finite order homeomorphism when restricted to $S - \mathcal{A}_\phi$ and $S - \mathcal{A}_\psi$, respectively.
- Passing to powers, may assume Φ, Ψ are the identity on $S - \mathcal{A}_\phi$, $S - \mathcal{A}_\psi$ respectively.
- Follows that Φ, Ψ are the identity on $S - B$.
- Working component-by-component, may assume B is connected.
- Easy case: B is an annulus, then it is a component of $\mathcal{A}_\phi$ and of $\mathcal{A}_\psi$, the restrictions are Dehn twists, and the rest is easy. (We have previously proved that in this case ϕ and ψ or ψ^{-1} have powers which agree).
- Easy case: B is a nonannulus isotopic to a component of $\mathcal{A}_\phi$ and of $\mathcal{A}_\psi$, and the restricted pseudo-Anosov homeomorphisms have the same stable-unstable lamination pair; the rest is easy.
- General case: B is a nonannulus, $\mathcal{A}_\phi, \mathcal{A}_\psi$ together fill B, and $\phi, \psi \mid B$ are not pseudo-Anosov with the same lamination pair.

The whole problem is therefore reduced to the following special case:

Lemma 2 (Filling Lemma)**.** *Suppose $\phi, \psi \in \mathcal{MCG}(S)$ are infinite order and the essential subsurfaces $\mathcal{A}_\phi, \mathcal{A}_\psi$ jointly fill S. Suppose furthemore that if ϕ, ψ are both pseudo-Anosov then they have different lamination pairs. Then there exist $m, n \geq 1$ such that $\phi^m \psi^n$ is pseudo-Anosov.*

This result is in the works of Ivanov and of McCarthy, and plays essentially the same technical role in their presentations that it is playing for us.

Their proof uses generalized source-sink dynamics on the Thurston boundary.

Our proof (Handel-M) will use laminations and Nielsen theory, and ideas from the Bestvina-Feighn-Handel proof of the Tits alternative for $\mathrm{Out}(F_n)$.

Quick idea of proof of the Filling Lemma:

- Let $\xi = \phi^m \psi^n$ (for appropriately chosen m, n).
- For every essential simple closed curve γ on S, we shall show
  - *either* "$\xi^k(\gamma)$ grows exponentially as $k \to +\infty$",
  - *or* "$\xi^{-k}(\gamma)$ grows exponentially as $k \to +\infty$".
- In either case, it is impossible that $\xi^k(\gamma) = \gamma$ for any k.
- Therefore the mapping class ξ must be pseudo-Anosov.
- The two "…grows exponentially…" phrases shall be defined in terms of attracting and repelling laminations.

Given $\phi \in \mathcal{MCG}(S)$,

$\Lambda^+(\phi)$ = the *attracting lamination* of ϕ = union of pseudo-Anosov unstable foliations on nonannulus components of $\mathcal{A}_\phi$ and union of core circles of annulus components of $\mathcal{A}_\phi$.

$\Lambda^-(\phi)$ = the *repelling lamination* of $\phi = \Lambda^+(\phi^{-1})$.

Note that the core circle of an annulus component of $\mathcal{A}_\phi$ is in both of the laminations $\Lambda^+(\phi)$, $\Lambda^-(\phi)$, and these are the only leaves in common.

Fix a hyperbolic structure on S.

Given a simple closed geodesic γ, we want to know:
how much of γ stays close to $\Lambda^+(\phi)$ or to $\Lambda^-(\phi)$?

Given:
Λ = a geodesic lamination Λ (such as $\Lambda^+(\phi), \Lambda^-(\phi)$)
γ = a simple closed geodesic
$L \geq 1$ a positive integer
$\epsilon > 0$

Define:
$N(\gamma, \Lambda; L, \epsilon)$ = maximal number of disjoint subsegments $\alpha \subset \gamma$ of length L such that each tangent vector on α is within distance ϵ of a tangent vector on Λ.

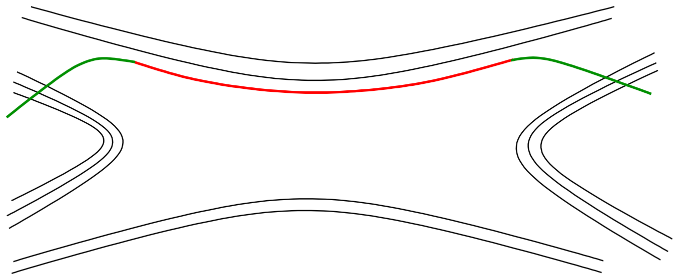

Figure 3: The black lines are leaves of the lamination Λ and the thick red-green curve is some simple closed curve on the surface. The red parts are the L-long subsegmants which are ϵ close to the lamination.

The "Stretching Lemma" says that as we iterate $\xi = \phi^m\psi^n$ on the curve γ, and we compute this number with respect to one of the two lamination $\Lambda^+(\phi)$ or $\Lambda^-(\psi)$, then for one of the two computations we see exponential growth.

Note: we are *not* directly proving exponential growth of the length of γ. This is an *a posteriori* consequence, but it is not the method of proof.

Lemma 3 (Stretching lemma)**.**
If $\phi, \psi \in \mathcal{MCG}(S)$ and $\mathcal{A}_\phi$, $\mathcal{A}_\psi$ fill S, then there exist $m, n > 0$ such that if $\xi = \phi^m\psi^n$, then there exists $L, \epsilon, A > 0$, $B > 1$ such that, for all simple closed geodesic $\gamma = \gamma_0$, defining γ_i to be the s.c. geodesic isotopic to $\xi^i(\gamma)$, there exists $\eta \in \{-1, +1\}$ such that one of the following holds:

- $N(\gamma_i; \Lambda^+_\phi, L, \epsilon) \geq AB^i$ *for* $i \geq 1$
- $N(\gamma_{-i}; \Lambda^-_\psi, L, \epsilon) \geq AB^i$ *for* $i \geq 1$

The Filling Lemma is an immediate consequence (because no γ is ξ periodic).

Outline of proof of Stretching Lemma:

- The two subsurfaces $\mathcal{A}_\phi, \mathcal{A}_\psi$ fill S, so γ intersects at least one of them.
- Let's look at cases to see how γ can be stretched:
  - A special case where γ under which $\xi = \phi^m\psi^n$ stretches γ.
  - A special case under which $\xi^{-1} = \psi^{-n}\phi^{-m}$
    stretches γ.
- We analyze what happens when these two special cases fail.

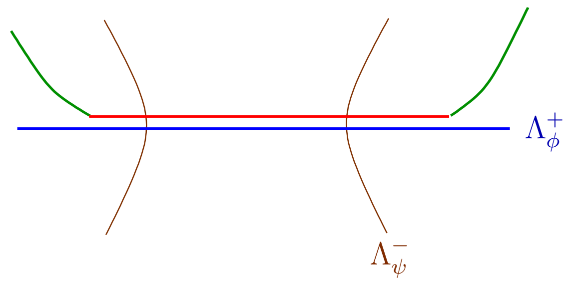

Figure 4: The red segment is γ' where γ is close to Λ^+_ϕ. It is not too close to Λ^+_ψ so it intersects some of its leaves.

- Special case: $N(\gamma, \Lambda^+_\phi; L, \epsilon) \geq 1$.

  Remark. One can fix ϵ to be the Margulis constant and only vary L in the proof to get the result.

- If $L >> 0$, then γ intersects $\mathcal{A}_\psi$ along a subarc γ' which is not too close to Λ^-_ψ (uses that $\mathcal{A}_\phi, \mathcal{A}_\psi$ fill) (Otherwise take $L \to \infty$ and get that Λ^+_ϕ and Λ^-_ψ share a leaf which is impossible for distinct minimal laminations, note that L, and the "not too close" constant depend on ϕ and ψ but not on γ).

- If $n >> 0$ then image under ψ^n of γ' has at least 2 segments α of length $\geq L$ within ϵ of Λ^+_ψ.(Here we make use of Neilsen Theory)

- If $L >> 0$ then each such α intersects $\mathcal{A}_\phi$ along a subarc α' which is not too close to Λ^-_ϕ.

- If $n >> 0$ then image under ϕ^m of β' has at least 2 segments γ of length $\geq L$ within ϵ of Λ^+_ϕ.

- Iterating this argument we conclude

  – If $N(\gamma_0, \Lambda^+_\phi; L, \epsilon) \geq 1$ then for $i \geq 1$ we have $N(\gamma_i, \Lambda^+_\phi; L, \epsilon) \geq 4^{i-1}$.

To make this argument rigorous, apply:

Theorem 4 (Nielsen Theory)**.** *Given $\psi \in \mathcal{MCG}(S)$ and component F of $\mathcal{A}_\psi$:*

- *If F is a nonannulus component, if Ψ represents ψ and is a pseudo-Anosov on F, if $x \in F$ is a fixed point with k stable and k unstable separatrices, if $\widetilde{\Psi}\colon \mathbf{H}^2 \to \mathbf{H}^2$ is a lift of Ψ fixing a lift $\tilde{x}$ of x, then the action of $\widetilde{\Psi}$ on $S^1_\infty = \partial\mathbf{H}^2$ has alternating source-sink dynamics with k sources and k sinks.*

- *If F is an annulus component, if Ψ represents ψ, fixes the core curve α of F, and twists each half of $F - \alpha$ at least once, if $\widetilde{\Psi}\colon \mathbf{H}^2 \to \mathbf{H}^2$ is a lift of Ψ fixing a lift $\tilde{\alpha}$ of α, then the action of $\widetilde{\Psi}$ on S^1_∞ has "shear dynamics" (see figure)*

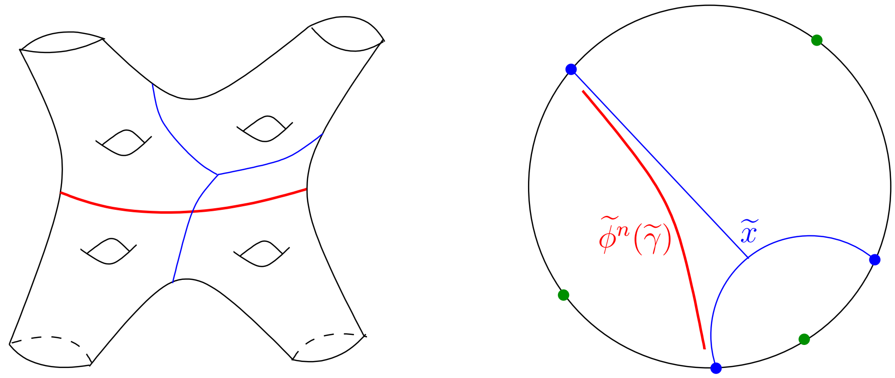

Figure 5: The blue points on $\partial\mathbf{H}^2$ are sinks for ϕ and the green points are sources for ψ. $\gamma \in N(\gamma_0, \Lambda^+_\phi; L, \epsilon) \geq 1$ then can uniformly bound the endpoints of $\widetilde{\phi}^n(\widetilde{\gamma})$ away from the fixed points of $\widetilde{\Lambda}^-_\psi$. Hence can find an m such that the endpoints of $\widetilde{\psi}^m(\widetilde{\phi}^n(\widetilde{\gamma}))$ are uniformly close to the attracting fixed points of $\widetilde{\psi}$

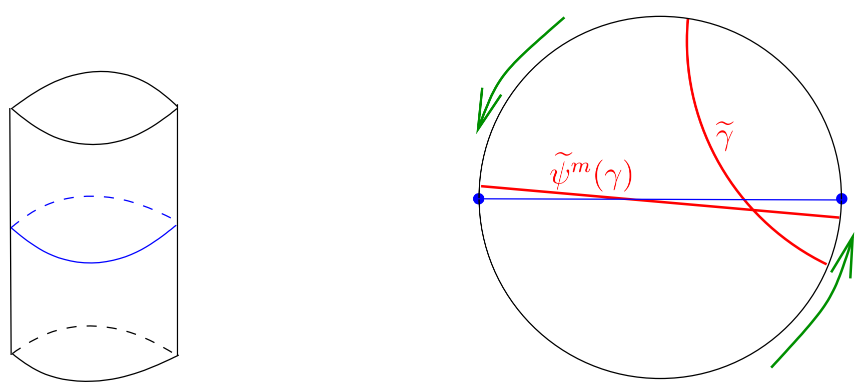

Figure 6: A Dehn twist has shear dynamics: contracts a half neighborhood of a fixed point and expands the other half. A large inough power of $\widetilde{\psi}$ will move $\widetilde{\gamma}$'s endpoints tward its fixed points.

Given a subarc γ' of γ as in the earlier argument, which is not too close to $\Lambda^-(\Psi)$ on F,
we obtain a lift $\widetilde{\gamma}$ somewhat close to $\tilde{x}$, but whose endpoints are not too close to sources of $\widetilde{\Psi}$.

Under a sufficiently large iterate,
the endpoints of $\widetilde{\Psi}^m(\widetilde{\gamma})$ are very close to sinks of $\widetilde{\Psi}$,
and so $\widetilde{\Psi}^m(\widetilde{\gamma})$ has a very long segment close to segments of $\widetilde{\Lambda}^+(\Psi)$.

We have now proved the Stretching Lemma in two special cases:

- $N(\gamma, \Lambda^+_\phi; L, \epsilon) \geq 1$ (what we just did)
- $N(\gamma, \Lambda^-_\psi; L, \epsilon) \geq 1$ (same argument applied to ξ^{-1}).

Remains to analyze what happens when these fail,
for example when

$$N(\gamma, \Lambda^-_\psi; L, \epsilon) = 0$$

- γ must intersect some component F of $\mathcal{A}_\psi$
- Since $N(\gamma, \Lambda^-_\psi; L, \epsilon) = 0$, γ is not too close to the "compressing lamination Λ^-_ψ.
- image under ψ^n of γ' has at least 1 segments α of length $\geq L$ within ϵ of Λ^+_ψ. (similar Nielsen theoretic argument)
- If $L >> 0$ then α intersects $\mathcal{A}_\phi$ along a subarc α' which is not too close to Λ^-_ϕ.
- If $n >> 0$ then image under ϕ^m of α'' has at least 2 segments γ of length $\geq L$ within ϵ of Λ^+_ϕ. (SAME Nielsen theoretic argument)

Conclusion:

- If Special Cases fail then $N(\gamma_0, \Lambda^-_\psi; L, \epsilon) = 0$
- $\implies N(\gamma_1, \Lambda^+_\psi; L, \epsilon) \geq 1$, which falls under the special case.