Mapping Class Groups
MSRI, Fall 2007
Day 3, September 19

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Today:

- Foundations of mapping class groups of finite type surfaces:
  - Definition of Teichmüller metric
  - Definition of pseudo-Anosov homeomorphisms
    - Both depend on concept of *singular xy-structures*
  - Thurston's classification of mapping classes
- Beginning of conjugacy classification of pseudo-Anosov mapping classes.

Definition of a finite type surface:

- An oriented surface S homeomorphic to a closed surface minus a finite subset.
- The closed surface is canonically homeomorphic to the *end compactification* of S.
- The missing points, or the corresponding ends of S, are called the *punctures* of S.
- Usually excluded are:
  - Spheres with ≤ 3 punctures — their mapping class groups are finite.

Mapping class group:

- $\mathcal{MCG}(S) = \mathrm{Homeo}_+(S)/\,\mathrm{Homeo}_0(S)$, where
  - $\mathrm{Homeo}_+(S)$ = group of orientation preserving homeomorphisms of S, with operation of composition

  - $\mathrm{Homeo}_0(S)$ = normal subgroup of homeomorphisms isotopic to the identity.

- For any set on which $\mathrm{Homeo}_+(S)$ acts
  (e.g. simple closed curves; conformal structures):
  - the orbits of the action of $\mathrm{Homeo}_0(S)$ are called *isotopy classes.*
  - action of $\mathrm{Homeo}_+(S)$ on given set descends to
    action of $\mathcal{MCG}(S)$ on set of isotopy classes
    (e.g. vertices of the curve complex; Teichmüller space)

Singular xy-structures.

- A *Euclidean structure* μ on a connected oriented surface F is a covering by charts with values in $\mathbf{E}^2$ and with overlap maps in $\mathrm{Isom}_+(\mathbf{E}^2)$.
- Associated to a Euclidean structure μ are:
  - the *holonomy homorphism*

$$h_\mu \colon \pi_1(F) \mapsto \mathrm{Isom}_+(\mathbf{E}^2)$$

  - the *rotational holonomy homomorphism*

$$r_\mu \colon \pi_1(F) \mapsto \mathrm{Isom}_+(\mathbf{E}^2) \to SO(2, \mathbf{R})$$

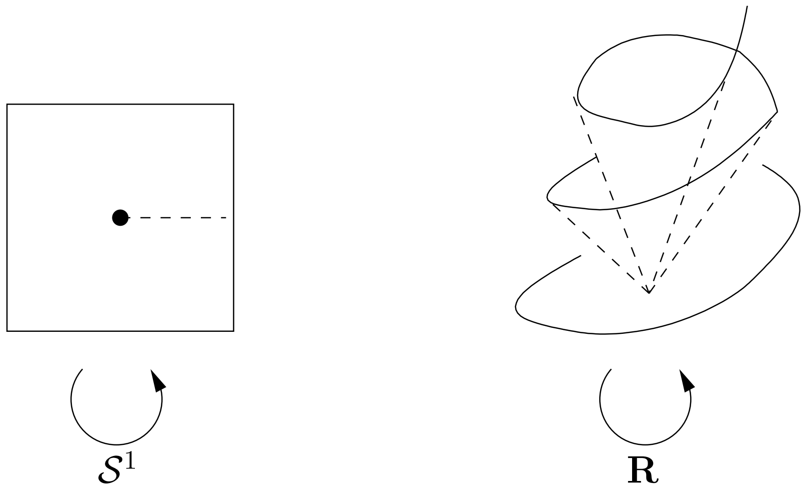
- A Euclidean structure μ is an *xy-structure* if

$$\mathrm{image}(r_\mu) \subset \{\pm \mathrm{Id}\}$$

  - Can change the atlas, by rotating charts, so that overlap maps take values in $\{\pm \mathrm{Id}\}$.
  - Overlap maps preserve the horizontal foliation of $\mathbf{E}^2$, and the transverse measure $|dy|$, inducing
    * the *horizontal measured foliation* $\mathcal{F}^h_\mu$ on S.
  - Also preserve vertical foliation and transverse measure $|dx|$, inducing
    * the *vertical measured foliation* $\mathcal{F}^v_\mu$ on S.
- Given an open disc D, $p \in D$, a Euclidean structure on $D - p$ has a *cone singularity of angle* $\alpha > 0$ at p if:
  - the metric completion at p is defined (i.e. by adding a single point to the end of $D - p$ corresponding to p)
  - the completion obtained by adding p is locally isometric to the completion of

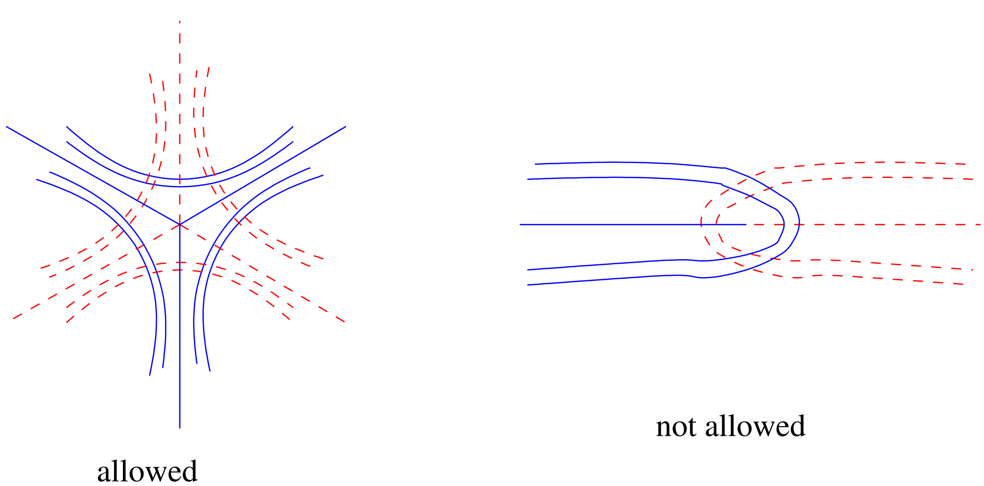
$$\widetilde{\mathbf{E}^2 - 0}/T_\alpha$$

    where T_α acts on $\widetilde{\mathbf{E}^2 - 0}$ by "rotation through α".



- A *singular xy-structure* on a finite type surface S consists of:
  - a finite set $\Sigma \subset S$
  - an xy-structure on $S - \Sigma$

  such that
  - each end of $S - \Sigma$ has a cone singularity, whose cone angles necessarily have the form $k\pi$ for integers $k \geq 1$
    - (no further condition on the cone angle at a puncture of S)
  - at each $s \in \Sigma$ the cone angle has the form $k\pi$ for $k \geq 3$.



Affine xy-maps. Given

- finite type surface S
- singular xy-structures σ_0, σ_1
- $h \in \mathrm{Homeo}_+(S)$

we say that h is an *affine xy map from σ_0 to σ_1* if:

- $h(\Sigma(\sigma_0)) = \Sigma(\sigma_1)$
- there exists a number $\lambda = \lambda(h) > 0$, called the *stretch factor* of h, such that in xy coordinate charts away from singularities, h looks locally like

$$\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) + (\text{translation})$$

Teichmüller space

- A conformal structure on a finite type surface is of *finite type* if the singularities at the punctures are removable.
- Teichmüller space $\mathcal{T}(S)$ = set of isotopy classes of finite type conformal structures on S, with the metric to be defined below.
- Example: given singular xy-structure μ on S with singularity set Σ,
  - charts of μ give conformal structure on $S - \Sigma$
  - Cone singularities are removable *(The transition maps near the singularities look like $z \longrightarrow z^{\frac{2}{k}}$ where $k \geq 3$ is the cone angle. The intersection of a coordinate chart containing the singularity with a cooridinate chart for some other point in that neighborhood is contained in $\frac{2}{k}$ of a circle, so the transition map is holomorphic there.)*
  - obtain finite type conformal structure on S
  - *wink wink* quadratic differentials ...

Theorem 1 (Teichmüller's Theorem)**.** *For any two non-isotopic finite type conformal structures σ_0, σ_1 on S there exist unique singular xy-structures μ_0, μ_1, homeomorphism $h \in \mathrm{Homeo}_0(S)$, and $\lambda = \lambda(\sigma_0, \sigma_1) > 0$ such that*

- *σ_i is the induced conformal structure of μ_i*
- *h is an affine xy-map from μ_0 to μ_1 with expansion factor λ.*

Moreover,

$$|\log(\lambda(\sigma_0, \sigma_1))|$$

is a metric on $\mathcal{T}(S)$ called the Teichmüller metric *(or maybe $\frac{1}{2}$ times this is the Teichmüller metric?).*

Note key restriction: h is isotopic to the identity.

Corollary 2 (Geodesics in Teichmüller space)**.**
$\mathcal{T}(S)$ is a geodesic metric space.
The "Teichmüller geodesic segment" from σ_0 to σ_1 is the path obtained from μ_0, μ_1 as above, altering the singular xy-structure μ_0 using the family of matrices $\left(\begin{smallmatrix} \ell & 0 \\ 0 & \ell^{-1} \end{smallmatrix}\right)$ for $\ell \in [1, \lambda]$ ("affine deformation" path)

Further facts about $\mathcal{T}(S)$:

- The Teichmüller geodesic segment between two points is unique.
- Each Teichmüller geodesic segment extends uniquely to a geodesic embedding $\mathbf{R} \hookrightarrow \mathcal{T}(S)$ ("affine deformation" lines).
- $\mathcal{MCG}(S)$ acts on $\mathcal{T}(S)$ by isometries.
- Action of $\mathcal{MCG}(S)$ on $\mathcal{T}(S)$ is faithful *except*:
  - if $S =$ torus with ≤ 2 punctures or closed surface of genus 2, in which case kernel $\approx \mathbf{Z}/2$.
- $\mathrm{Mod}(S) =$ image of the action of $\mathcal{MCG}(S)$ on $\mathcal{T}(S)$. So:
  - $\mathrm{Mod}(S) \approx \mathcal{MCG}(S)$ or $\mathcal{MCG}(S)/(\mathbf{Z}/2)$
- Royden's Theorem: $\mathrm{Mod}(S) = \mathrm{Isom}_+(\mathcal{T}(S))$.

pseudo-Anosov mapping classes. A homeomorphism $\Phi \colon S \to S$ is *pseudo-Anosov* if there exists a singular xy-structure μ and a number $\lambda = \lambda(f) > 1$ such that Φ is an xy-affine map from μ to μ with stretch factor λ.

- The *unstable foliation* of Φ is
$$\mathcal{F}^u = \mathcal{F}^h$$
whose leaves are stretched by factor of λ.
- The *stable foliation* of Φ is
$$\mathcal{F}^s = \mathcal{F}^v$$
whose leaves are compressed by factor of λ.

A mapping class $\phi \in \mathcal{MCG}(S)$ is *pseudo-Anosov* if it is represented by a pseudo-Anosov homeomorphism.

Note: By definition (and by Teichmüller's theorem), a pseudo-Anosov mapping class has an axis in Teichmüller space, a geodesic along which it translates.

Reducible mapping classes.

- An *essential curve* on S is a simple closed curve which does not bound a disc with ≤ 1 punctures.
- An *essential curve system* on S is a disjoint union of essential curves.
- $\mathcal{MCG}(S)$ acts on the set of isotopy classes of essential curve systems.
- A mapping class $\phi \in \mathcal{MCG}(S)$ is *reducible* if
  - ϕ fixes the isotopy class of some essential curve system
  - equivalently, ϕ is represented by $\Phi \in \mathrm{Homeo}_+(S)$ that preserves some essential curve system.

Theorem 3 (Thurston)**.** *For every mapping class $\Phi \in \mathcal{MCG}(S)$ one of the following occurs:*

- Φ *is of finite order,*
- Φ *is reducible,*
- Φ *is pseudo-Anosov*

Note: The only overlap of these cases is between finite order and reducible.

Outline of Bers' Proof: Define the *translation number* of Φ acting on $\mathcal{T}(S)$ as

$$t(\Phi) = \inf_{\sigma \in \mathcal{T}(S)} d(\sigma, \Phi(\sigma))$$

Break into cases, depending on whether the infimum defining $t(\Phi)$ is realized as a minimum.

Case 1: $t(\Phi)$ is realized.

Case 1a: $t(\Phi) > 0$. Then Φ is pseudo-Anosov: unravelling the definitions it follows immediately that Φ is represented by a pseudo-Anosov homeomorphism of S.

Case 1b: $t(\Phi) = 0$. Then Φ has a fixed point in $\mathcal{T}(S)$, and unravelling the definitions it follows immediately that there is a representative ϕ of Φ and a conformal structure σ such that $\phi(\sigma) = \sigma$. But the group of conformal automorphims of σ is finite.

Case 2: $t(\Phi)$ is not realized. Then Bers' constructs an essential curve system invariant under Φ. (This is the hard work in Bers' proof).

$\diamondsuit$

Remark:

- Proof shows that every finite order mapping class is represented by a finite order homeomorphism.
- If $t(\Phi) > 0$ is not realized then Φ has a pseudo-Anosov subsurface.
- If $t(\Phi) = 0$ is not realised the Φ is a composition of a dehn twist about a system of curves and a finite order symmetry of the surface.

Contrast with Thurston's proof:

- Construct a compactification $\overline{\mathcal{T}}(S)$ of $\mathcal{T}(S)$:
  - $\overline{\mathcal{T}}(S) \approx B^{6g-6+2p}$
  - $\mathcal{T}(S)$ is the interior of the ball
  - The boundary $\partial\overline{\mathcal{T}}(S)$ is identified with the space of *projective measured foliations (or laminations)* $\mathbf{P}\mathcal{M}F(S) = \mathbf{P}\mathcal{M}L(S)$ (more on this later).
- Given $\Phi \in \mathcal{MCG}(S)$, action of Φ has a fixed point in $\overline{\mathcal{T}}(S)$ (Brouwer F.P.T.)
- Break into cases:
  - Fixed point in $\mathcal{T}(S)$: Φ has finite order
  - Fixed point in $\mathbf{P}\mathcal{M}F(S)$: use it to show that Φ is reducible, pseudo-Anosov, or finite order.

First step in constructing conjugacy invariants:

Theorem 4 (Uniqueness of pseudo-Anosov homeomorphisms)**.** *Each pseudo-Anosov mapping class on S is represented by a* unique *pseudo-Anosov homeomorphism* up to topological conjugacy.

More precisely, given two pseudo-Anosov homeomorphisms $\Phi_0, \Phi_1 \in \mathrm{Homeo}_+(S)$ *in the same mapping class, there exists* $\Psi \in \mathrm{Homeo}_0(S)$ *such that*

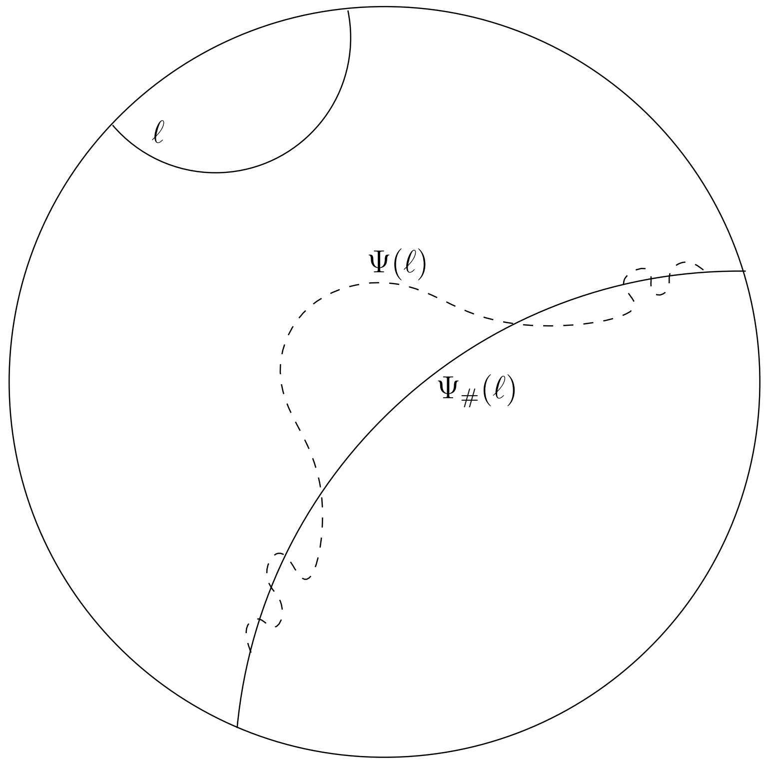
$$\Phi_1 = \Psi\Phi_0\Psi^{-1}$$

Thus: to classify pseudo-Anosov mapping classes up to conjugacy, it suffices to classify pseudo-Anosov homeomorphisms up to topological conjugacy. *(If two homeomorphisms represent conjugate mapping classes then there is a homeomorphism of the surface which conjugates between them, not just up to isotopy).*

Note: Theorem says pseudo-Anosov homeomorphisms in the same mapping class are unique up to "conjugacy by isotopy".

Proof when S is closed: Define *leaf* of a singular foliation in a way that allows it to pass arbitrarily through a singularity.

Step 1: Leaves of $\widetilde{\mathcal{F}}^s$ and of $\widetilde{\mathcal{F}}^u$ in $\widetilde{S} \approx \mathbf{H}^2$ are uniformly quasigeodesic.



- The metric μ on S is locally CAT(0).
- Therefore, the metric $\widetilde{\mu}$ on $\widetilde{S}$ is globally CAT(0).
- Leaves of $\widetilde{\mathcal{F}}^s$ and $\widetilde{\mathcal{F}}^u$ are μ local geodesics.
- Therefore, those leaves are globally geodesic.
- $\widetilde{\mu}$ is K, C quasi-isometric to $\mathbf{H}^2$, for some $K \geq 1$, $C \geq 0$.
- Therefore, leaves are K, C quasigeodesics.

Comment on punctured case: Step 1 is false as stated. But if you first push leaves of $\mathcal{F}^s$ and $\mathcal{F}^u$ away from the cusps, then it becomes true.

This makes the proof in the punctured case more complicated, but it is basically the same proof.

Preparation for step 2.
For any quasi-isometry $\Psi\colon \mathbf{H}^2 \to \mathbf{H}^2$
and any geodesic $\ell \subset \mathbf{H}^2$:

- let $\Psi_{\#}\ell =$ the geodesic that fellow travels $\Psi(\ell)$,
- let $\Psi_\ell\colon \ell \to \Psi_{\#}\ell$ be the composition of $\Psi \bigm| \ell$ followed by closest point projection to ℓ.

- Let $\Psi_\ell^n \colon \ell \to \Psi_\#^n \ell$ denote n iterations of Ψ_ℓ, namely,

$$\Psi_\ell^n = \Psi_{\Psi_\#^{n-1}(\ell)} \circ \cdots \circ \Psi_{\Psi_\#(\ell)} \circ \Psi_\ell$$

- say that Ψ *coarsely contracts* ℓ if, under iteration of Ψ_ℓ, distance eventually falls below a uniformly finite threshold. To be precise:
  there exists $A \geq 0$ such that for all $x, y \in \ell$ there exists N such that for all $n \geq N$,
  $$d(\Psi_\ell^n(x), \Psi_\ell^n(y)) \leq A$$

Let

- $\partial^2 \pi_1 S = \partial \pi_1 S \times \partial \pi_1 S - \Delta$
- $\partial^2 \widetilde{\mathcal{F}}^s = \{(\xi, \eta) \in \partial^2 \pi_1 S \bigm| \xi, \eta \text{ are the endpoints of some leaf of } \widetilde{\mathcal{F}}^s\}$
- $\partial^s \widetilde{\mathcal{F}}^u \subset \partial^2 \pi_1 S$ is similarly defined.

Recall notation:

- ϕ is a pseudo-Anosov mapping class,
- Φ is a pseudo-Anosov homeomorphism representing ϕ
- $\mathcal{F}^s, \mathcal{F}^u$ are the stable and unstable foliations representing ϕ.

Step 2: Each $(\xi, \eta) \in \partial^2 \widetilde{\mathcal{F}}^s$ satisfies the following property: given

- any hyperbolic structure on S,
- any $\Phi' \in \mathrm{Homeo}_0(S)$ representing ϕ,
- any lift $\widetilde{\Phi}' \colon \mathbf{H}^2 \to \mathbf{H}^2$,

$\widetilde{\Phi}'$ coarsely contracts the geodesic $\overline{\xi, \eta}$.

Proof of Step 2. Let $\ell = \widetilde{\mathcal{F}}^s(\xi, \eta)$ be the leaf of $\widetilde{\mathcal{F}}^s$ with endpoints ξ, η, which fellow travels $\ell' = \overline{\xi, \eta}$.

Iteration of $\widetilde{\Phi}'_{\ell'}$ on $x' \in \ell'$ is tracked up to a uniformly finite distance iteration of the following process in ℓ:

- apply Φ to a point in the stable leaf ℓ to get a point in the stable leaf $\Phi(\ell)$, then perturb a uniformly bounded distance to another point in $\Phi(\ell)$.

The latter process obviously decreases distances until they fall below some uniformly finite threshold.

$\diamondsuit$

TO BE CONTINUED NEXT TIME ...