Mapping Class Groups
MSRI, Fall 2007
Day 7, October 18

November 4, 2007

Finite order mapping classes.

Theorem 1. *Every finite order mapping class $\phi \in \mathcal{MCG}(S)$ of a finite type surface (finite genus and number of punctures) is represented by a finite order homeomorphism $\Phi \in \mathrm{Homeo}_+(S)$.*

- First proof by Nielsen.
- Error in Nielsen's proof found, and corrected, by Zieschang.
- Nielsen's proof fails when (a posteriori) the quotient orbifold S/Φ is a sphere with 3 cone points.
- Nowadays, almost any proof of Thurston's classification can be refined to give a proof of this theorem.

Proof by refinement of Bers' proof: Let

$$\begin{aligned} t(\phi) &= \text{translation distance of } \phi \text{ on } \mathcal{T}(S) \\ &= \inf_{\sigma \in \mathcal{T}(S)} d(\sigma, \phi(\sigma)) \end{aligned}$$

Recall case analysis:

- $t(\phi) = 0$ OR $t(\phi) > 0$
- The infimum is a minimum OR
  the infimum is not a minimum.
- Case 1: $t(\phi) = 0$ and the infimum is a minimum.
  - $\implies \exists \sigma \in \mathcal{T}(S)$ such that $\phi(\sigma) = \sigma$
  - Pick Σ = conformal structure on S representing σ

– Pick Φ' = homeomorphism on S representing ϕ

– $\Phi'(\Sigma)$ is isotopic to Σ
$\implies \exists \Psi \in \mathrm{Homeo}_0(S)$ such that $\Phi'(\sigma) = \Psi(\sigma)$

– $\Phi = \Psi^{-1}\Phi'$ represents ϕ.

– $\Phi(\sigma) = \sigma$.

– Recall that the group of conformal automorphisms of any conformal structure is finite.

– $\implies \Phi$ is a finite order homeomorphism.

Remark. A proof that the group of conformal automorphisms is finite when S is compact (this is true for any surface of finite type):
By the uniformization theorem, we may prove that the isometry group of a hyperbolic surface is finite. First notice that the group of continuous maps from S to itself (with the sup metric) is compact because every sequence has a convegent subsequence (Arzela-Ascoli). A limit of a sequence of isometries is an isometry and therefore the subspace of isometries is compact. We will show that it is also discrete, and therefore finite. There is an ε such that if $d(f, \mathrm{id}) < \varepsilon$ then f is homotopic to the identity. The lift of f to $\mathbf{H}^2$ is a mobius transformation which stays a bounded distence from the identity hence induces the identity map on $\partial\mathbf{H}^2$. But a mobius transformation which fixes three points or more is the identity.

In all other cases, the mapping class ϕ has infinite order:

- Case 2: $t(\phi) = 0$ and is not realized.
  – $\exists k \geq 1$ such that ϕ^k = nontrivial product of nontrivial powers Dehn twists about disjoint curves $\implies \phi$ has infinite order.
- Case 3: $t(\phi) > 0$ and is realized.
  – $\implies \phi$ is pseudo-Anosov
  $\implies \phi$ has infinite order.
- Case 4: $t(\phi) > 0$ and is not realized.
  – $\implies \exists k \geq 1$ such that ϕ^k has an invariant subsurface on which it is pseudo-Anosov,
  $\implies \phi$ has infinite order.

Remark. in all of the cases $2-4$ we can find a simple closed curve whose iterates under ϕ are all distinct.

♢

Homological consequence of finite order

Theorem 2. *If a non-trivial $\phi \in \mathcal{MCG}(S)$ has finite order then action of ϕ on $H_1(S; \mathbf{Z}/3)$ is nontrivial.*

Proof. Adapt proof from Casson-Bleiler that action on $H_1(S;\mathbf{Z})$ is nontrivial:

- Choose finite order representative Φ of ϕ.
- Choose Φ-invariant cell decomposition $\mathcal{C}$.
  If S is a closed surface, lift a cell decomposition of S/Φ. If S is a punctured surface, lift a cell decomposition of a spine of the S/Φ.
- Use cellular chain complex of $\mathcal{C}$ (with $\mathbf{Z}$ coefficients in C-B).
- Find a 1-cycle c (with $\mathbf{Z}$ coefficients in C-B) such that $\Phi(c)$ is not homologous to c.
- Can ignore the absolute values of coefficients in this proof, and use only their signs.
- Coefficient can have one of three signs: $-$, 0, $+$.
- Use $\mathbf{Z}/3$ to represent sign.
- Check that the arithmetic of the proof still works.

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Corollary 3. *Every finite subgroup of* $\mathrm{Homeo}_+(S)$ *embeds in* $\mathcal{MCG}(S)$.

Given a conformal structure Σ on S, its conformal automorphism group $\mathrm{Aut}(\Sigma)$ is a finite group of homeomorphisms, and so:

Corollary 4. *For each $\sigma \in \mathcal{T}(S)$ and each conformal structure Σ representing σ,*

- $\mathrm{Aut}(\Sigma)$ *embeds in* $\mathcal{MCG}(S)$
- *image of this embedding* $= \mathrm{Stab}(\sigma) =$ *subgroup of $\mathcal{MCG}(S)$ that stabilizes σ.*

Corollary 5. *$\mathcal{MCG}(S)$ has a torsion free subgroup of finite index, namely the kernel of the homomorphism*

$$\mathcal{MCG}(S) \to \mathrm{Aut}\big(H_1(S;\mathbf{Z}/3)\big)$$

Fixed sets in Teichmüller space. Let

$$\begin{aligned}
\phi &= \text{a finite order element of } \mathcal{MCG}(S) \\
\Phi &= \text{a finite order homeomorphism representing } \phi \\
S/\Phi &= \text{the quotient orbifold} \\
\mathcal{T}(S/\Phi) &= \text{the Teichmüller space of } S/\Phi \\
&= \{\text{Hyperbolic structures with cone angles}\} \Big/ \text{ isotopy rel cone points} \\
&= \{\text{Conformal structures}\} \Big/ \text{ isotopy rel cone points}
\end{aligned}$$

Remark. Using the discreteness of the action of $\pi_1 S$ on $\mathbf{H}^2$ one can show that there are only finitely many conepoints for a given finite order Φ.

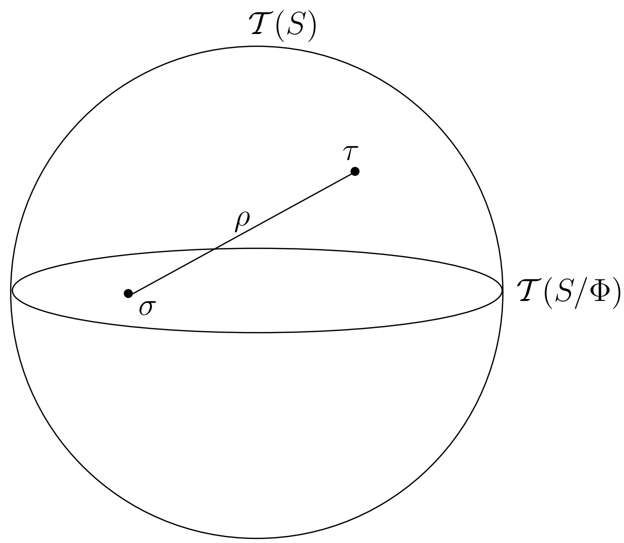

Figure 1: For the proof of theorem 6: $\sigma \in \mathcal{T}(S/\Phi)$ and τ is fixed by ϕ. ρ is the geodesic between them and μ the xy structure corresponding to it.

Lifting structures defines an isometric embedding

$$\mathcal{T}(S/\Phi) \hookrightarrow \mathcal{T}(S)$$

Remark. The isometric part comes from Teichmüller's theorem.

Theorem 6. *The fixed point set of the action of ϕ on $\mathcal{T}(S)$ is the image of the embedding $\mathcal{T}(S/\Phi) \hookrightarrow \mathcal{T}(S)$.*

Proof. Obviously the points in the image are fixed. For the converse, let:

- $\tau \in \mathcal{T}(S)$ be a fixed point.
- $\sigma \in$ image of $\mathcal{T}(S/\Phi) \hookrightarrow \mathcal{T}(S)$, also fixed.
- Σ be a conformal structure on S representing σ such that $\Phi(\Sigma) = \Sigma$.
- $\rho\colon [0, +\infty) \to \mathcal{T}(S)$ is the geodesic ray from $\sigma = \rho(0)$ through τ.
- $\mu =$ the unique xy-structure on Σ that generates the ray ρ (applying Teichmüller's Theorem).
- Uniqueness (and naturality) of geodesics $\implies$
  each point of ρ is fixed by ϕ $\implies$
  the xy structure μ is fixed by Φ $\implies$
  μ descends to an xy-structure μ/Φ on S/Φ.

- Ray in $\mathcal{T}(S/\Phi)$ generated by μ/Φ lifts to the ray in $\mathcal{T}(S)$ generated by μ
- $\implies \tau$ is in the image of $\mathcal{T}(S/\Phi)$.

$\diamondsuit$

Reducing $\mathcal{MCG}$ conjugacy to topological conjugacy

Theorem 7. *If $\Phi, \Phi' \in \text{Homeo}_+(S)$ are two finite order homeomorphisms in the same mapping class then there exists $\Psi \in \text{Homeo}_0(S)$ such that $\Phi' = \Psi\Phi\Psi^{-1}$.*

Corollary 8. *If $\phi, \phi' \in \mathcal{MCG}(S)$ are finite order mapping classes represented by finite order homeomrophisms $\Phi, \Phi' \in \mathcal{MCG}(S)$ then ϕ, ϕ' are conjugate in $\mathcal{MCG}(S)$ if and only if Φ, Φ' are topologically conjugate by an element of* $\text{Homeo}_+(S)$. $\diamondsuit$

Proof of Theorem.

$$\begin{aligned} &\text{Fixed set of} \quad \phi \quad \text{in} \quad \mathcal{T}(S) \\ = \quad &\text{image of} \quad \mathcal{T}(S/\Phi) \hookrightarrow \mathcal{T}(S) \\ = \quad &\text{image of} \quad \mathcal{T}(S/\Phi') \hookrightarrow \mathcal{T}(S) \end{aligned}$$

Pick $\sigma \in \mathcal{T}(S)$ in the fixed set of ϕ.
Pick conformal structures Σ, Σ' representing σ such that

$$\Phi(\Sigma) = \Sigma$$
$$\Phi'(\Sigma') = \Sigma'$$

Pick $\Psi \in \text{Homeo}_0(S)$ such that $\Psi(\Sigma) = \Sigma'$.

$$\implies \Phi^{-1}\Psi^{-1}\Phi'\Psi(\Sigma) = \Sigma$$

But $\Phi^{-1}\Psi^{-1}\Phi'\Psi \in \text{Aut}(\Sigma)$ is isotopic to the identity.
And $\text{Aut}(\Sigma)$ embeds in $\mathcal{MCG}(S)$.
$\implies \Phi^{-1}\Psi^{-1}\Phi'\Psi$ *equals* the identity in $\text{Aut}(\Sigma)$
$\implies \Phi = \Psi^{-1}\Phi'\Psi$.

$\diamondsuit$

Conjugacy invariants of finite order homeomorphisms
For simplicity, assume S is closed — no punctures.
Let $\Phi \in \text{Homeo}_+(S)$ have finite order,

$$\begin{aligned} o(\Phi) &= \text{order of } \Phi \\ &= \text{smallest } K \text{ such that } \Phi^K \text{ equals Id} \end{aligned}$$

S/Φ is a closed, oriented 2-orbifold

= closed oriented surface with finitely many cone points labelled by $\mathbf{Z}/n$, $n \geq 2$.
$\forall x \in S$, let

$$o(x) = \text{order of } x$$
$$= \text{smallest } k \geq 1 \text{ such that } \Phi^k(x) = x$$

$$r(x) = \text{rotation number of } x$$
$$= \text{rational number in } [0,1) \text{ such that}$$
$$\Phi^k \text{ rotates tangent plane at } x$$
$$T(x) \text{ through angle } 2\pi r(x)$$

$x \in S$ is a *proper periodic point* if $o(x) < o(\Phi)$.

Remark. In general case with punctures, treat them like a special subclass of proper periodic points; by convention should even allow the possibility that $o(\text{puncture}) = o(\Phi)$

$o(x)$ is bounded.
$r(x)$ has bounded numerators and denominator.
$\#\{\text{proper periodic point of } \Phi\}$ is bounded S.

These bounds depend only on the topology of S (not on Φ).

Define $N_\Phi(o, r) = \#$ proper periodic point x of Φ such that

$$o(x) = o$$
$$r(x) = r$$

There are only finitely many possible functions N_Φ, in terms of topology of S.

Theorem 9 (Nielsen)**.**
$\forall$ *finite order* $\Phi, \Phi' \in \text{Homeo}_+(S)$*, TFAE:*

- $N_\Phi = N_{\Phi'}$
- $\exists \Psi \in \text{Homeo}_+(S)$ *such that* $\Phi = \Psi^{-1}\Phi'\Psi$

Special case: $N \equiv 0$**.** That is, there are *no* proper periodic points
Equivalently, action of Φ is properly discontinuous.
Equivalently, S/Φ is a surface and the quotient map $S \to S/\Phi$ is a covering map.
Must prove: for each K there exists up to topological conjugacy a *unique* properly discontinuous action of $\mathbf{Z}/K$ on S.

Special special case: There exists up to topological conjugacy a *unique* properly discontinuous action of $\mathbf{Z}/2$ on S_3.

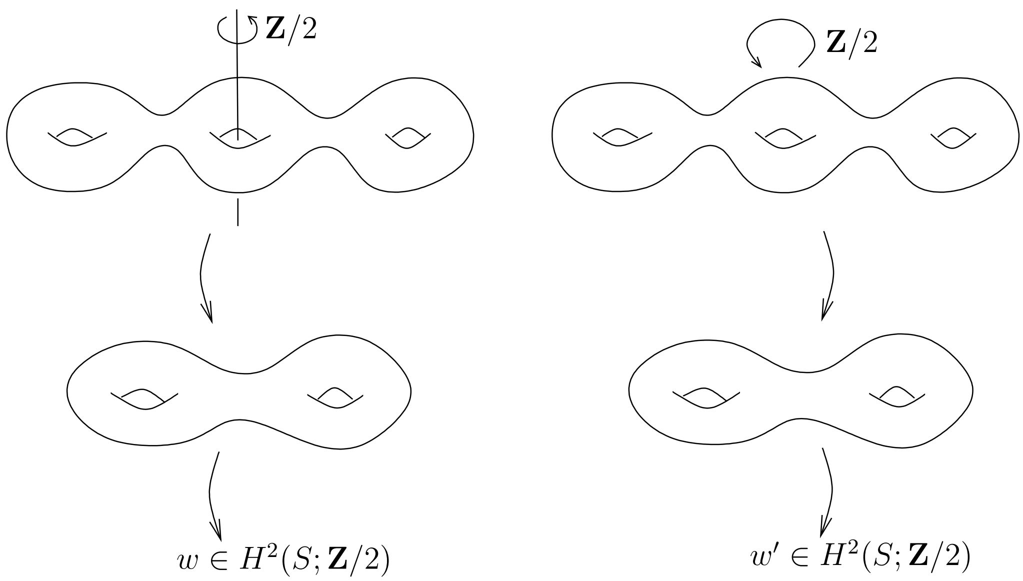

Figure 2: Special special case

- $\forall$ prop. disc. action of $\mathbf{Z}/2$ on S_3,

  $S_3/(\mathbf{Z}/2) \approx S_2$ (Euler characteristic calculation)

- Lifting theorem $\implies$ action determined (up to top. conj.) by a homomorphism

$$\pi_1(S_2) \mapsto \mathbf{Z}/2$$

  determined in turn by an element

$$\omega \in H^1(S_2; \mathbf{Z}/2)$$

- $\mathcal{MCG}(S_2)$ acts transitively on nonzero elements of $H^1(S_2; \mathbf{Z}/2)$ (get hands dirty)
- $\implies$ any two actions of $\mathbf{Z}/2$ on S_3 are topologically conjugate.

Back to ordinary special case:
S is closed (no punctures), action of $\mathbf{Z}/K$ is properly discontinuous.

- $\forall$ prop. disc. actions of $\mathbf{Z}/K$ on S, the quotient surface S' is chacterized by

$$\chi(S') = \chi(S)/K$$

- Lifting theorem $\implies$ action is determined (up to top. conj.) by a surjective homomorphism

$$\pi_1(S') \mapsto \mathbf{Z}/K$$

- Surjective homomorphisms $\pi_1(S') \mapsto \mathbf{Z}/K$ correspond bijectivelly to *primitive* elements of $H^1(S;\mathbf{Z}/K)$.

- $\mathcal{MCG}(S')$ acts transitively on primitive elements of $H^1(S;\mathbf{Z}/K)$.

General case: Bore out neighborhoods of proper periodic points.
On the complement, the action is properly discontinuous.
Apply covering space theory on surfaces with boundary.
Topological conjugacy on boundary is already given (by rotation number).
Must extend the topological conjugacy from the boundary to the whole surface.
Apply relative version of above argument (rel boundary).
Works in punctured case too, treating punctures as a special class of proper periodic points