Homework 1: Mobius transformations and the upper half plane model

By $\hat{\mathbb{C}}$ we denote the set $\mathbb{C} \cup \{\infty\}$ with the standard topology (a.k.a. the Riemann sphere, or the real projective plane).

Definition 1. A fractional linear transformation, or a Mobius transformation is a map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form $f(z) = \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. (Here we only consider the orientation preserving transformations).

Definition 2. A generalized circle in $\mathbb{C}$ is a circle or a line (both correspond to circles on $\hat{\mathbb{C}}$).

1. Prove the following properties of Mobius transformations:
   (a) f is a homeomorphism.
   (b) The map $\phi : \mathrm{SL}(2, \mathbb{C}) \to \mathrm{Mob}$ defined by $\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{az+b}{cz+d}$ is an epimorphism. Find its kernel.
   (c) Show that any Mobius transformation f is either affine or can be written as $f(z) = h(g(k(z)))$ where h, k are affine maps and $g(z) = \frac{1}{z}$. (A map is affine if it has the form $az + b$).
   (d) Show that Mobius transformations preserve generalized circles.
   (e) For every 4-tuple of distinct points in $\mathbb{C}$ define the cross ratio as

   $$[x : y : z : w] = \frac{(x - w)(z - y)}{(x - y)(z - w)}$$

   This function extends continuously to 4-tuples in $\hat{\mathbb{C}}$. Prove that Mobius transformations preserve the cross ratio.
   (f) Show that the group of Mobius transformation acts transitively on triples in $\hat{\mathbb{C}}$. That is, for every six distinct points $x, y, z, x', y', z' \in \hat{\mathbb{C}}$ there is a unique Mobius transformation f so that

   $$f(x) = x' \quad f(y) = y' \quad f(z) = z'$$

2. Compute the hyperbolic distance from $2i$ to $1 + 2i$.

3. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function, prove that for any vector $v \in T_w\mathbb{C}$, with $v = \begin{pmatrix} a \\ b \end{pmatrix}$ we have

$$D_w f \cdot v = \frac{\partial f}{\partial z}(w) \cdot (a + ib)$$

Where the multiplication on the left is matrix multiplication and on the right it is just multiplication in $\mathbb{C}$.
This basically says that if f is holomorphic, then instead of taking its derivative as a function in two variables, we can take its derivative with respect to z as a one variable function. (Cool!)

4. Show that the map $f(x, y) = i\sqrt{x^2 + y^2}$ is contracting with respect to the hyperbolic metric. (Note that we cannot use the result in the previous problem since f is not holomorphic).