Homework 3: The disc model

The disc model of the hyperbolic plane consists of the set

$$\mathbb{D} = \{z \mid |z| < 1\}$$

And an inner product $(u,v)_z = \frac{4}{(1-|z|^2)^2}(u,v)_{\mathbb{E}}$. In Anderson 120-121 you can find a proof that the resulting path metric $d_{\mathbb{D}}$ is the push-forward of the metric in (U, d_U) under the map $\xi : U \to \mathbb{D}$, $\xi(z) = \frac{iz+1}{-z-i}$.

1. Show that the Mobius transformations which preserve the unit disc have the form $\frac{\alpha z+\beta}{\bar{\beta}z+\bar{\alpha}}$ for some $\alpha, \beta \in \mathbb{C}$. This group is denoted $Mob(\mathbb{D})$).

2. Prove that elements in $Mob(\mathbb{D})$ are isometries of $(\mathbb{D}, d_{\mathbb{D}})$. Hint: show that $\frac{|f(z)|}{im(f(z))} = \frac{2}{1-|z|^2}$

3. Find the hyperbolic length of the following paths:

   (a) $p(t) = t$, $p : [0,r] \to \mathbb{D}$.

   (b) $q(t) = s(\cos t, \sin t)$, $q : [0, 2\pi] \to \mathbb{D}$ for some $s \in [0,1]$

   What is the the length of a hyperbolic circle as a function of its hyperbolic radius?

4. What is the sphere of radius 1 about 0? i.e. describe $S(0,1) = \{z | d_{\mathbb{D}}(0,z) = 1\}$. In general, what are hyperbolic circles about points?

5. Let Δ be a triangle with angles α, β, γ and whose opposite sides have lengths a, b, c respectively. Prove the following:

   (a) The hyperbolic law of sines:

   $$\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma}$$

   (b) The first hyperbolic law of cosines:

   $$\cosh(a) = \cosh(b)\cosh(c) - \sinh(c)\sinh(b)cos(\alpha)$$

   (c) The second hyperbolic law of cosines:

   $$\cos(\gamma) = -\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)\cosh(c)$$

   Note that the second hyperbolic law of cosines implies that knowing the angles of a triangle determines its side lengths.

(Hint: pages 181-185 in Anderson).

6. Given α, β, γ and two triangles Δ, Δ' with interior angles α, β, γ. Show that there is ahyperbolic isometry taking Δ to Δ'.

7. Given $a, b, c > 0$ show that a hexagon with alternating side lengths a, b, c is unique up to isometry.