Homework 5: Riemann Surfaces.

1. Consider $\hat{\mathbb{C}}$ with the following coordinate neighborhoods: $U_0 = \mathbb{C}, \phi_0(z) = z$, $U_\infty = \mathbb{C} \setminus 0$ and $\phi_\infty(z) = \frac{1}{z}$.

   (a) Show that this is a Riemann Surface.

   (b) Let X denote $\hat{\mathbb{C}}$ with another Riemann structure given by $\phi_0(z) = \bar{z}$ and $\phi_\infty(z) = \frac{1}{\bar{z}}$. Prove that this is a Riemann structure and that it is biholomorphically equivalent to $\hat{\mathbb{C}}$.

   (c) Is $\hat{\mathbb{C}}$ biholomorphically equivalent to $(\mathbb{C}, \mathrm{id})$ or to D the unit disc in $\mathbb{C}$ with the inclusion as a coordinate neighborhood? Is $\mathbb{C}$ equivalent to D?

2. Consider R the region in $\mathbb{C}$ bounded by the following geodesics: $[0, 2], [2, 2+i], [2 + i, 1 + i], [1 + i, 1 + 2i], [1 + 2i, 2i], [2i, 0]$. Let M be the quotient surface obtained by identifying parallel sides of R. For example, $[0, i]$ is identified with $[2, 2 + i]$ via the map $f(z) = z + 2$, and $[i, 2i]$ is identified with $[1 + i, 1 + 2i]$ by the map $g(z) = z + 1$ etc

   (a) By the classification of surfaces, M is homeomorphic to either a sphere, a connected sum of n tori, or a connected sum of n real projective planes. Which surface is M?

   (b) What points of R are identified with 0 under this gluing?

   (c) What does a neighborhood of $\frac{i+1}{2}$ in M look like? Describe a neighborhood of $\frac{1}{2}$ and a neighborhood of 0 in M.

   (d) Prove that M is a Riemann Surface. That is, find charts (U_j, ϕ_j) which are homeomorphisms from U_j to some open subset D_j in $\mathbb{C}$. Check that the transition maps are holomorphic.

3. Let R be a Riemann surface and $\tilde{R}$ a topological cover of R.

   (a) Show that there is a conformal structure on $\tilde{R}$ so that the covering map $p : \tilde{R} \to R$ is holomorphic.

   (b) Recall that $\pi_1(R)$ acts on $\tilde{R}$ by covering translations. Prove that if $t_\gamma : \tilde{R} \to \tilde{R}$ is a covering translation then it is a biholomorphic map (holomorphic with a holomorphic inverse).